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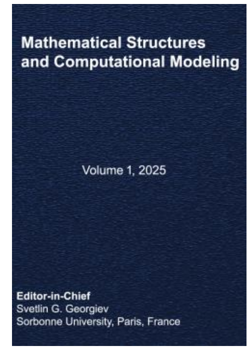

A Study of a Generalized Logarithmic–Hyperbolic Class of Integrals

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ABSTRACT

We introduce a generalized class of definite integrals

$$I[f] = \int_{-\infty}^{\infty} \frac{f(ln(1+x^2))}{cosh\, x} dx,$$

where f is real- or complex-valued and satisfies mild growth conditions ensuring convergence. We obtain closed-form evaluations for several families of f, including polynomial/logarithmic choices, power-type families, and trigonometric functions of $ln(1+x^2)$. Full proofs of the intermediate lemmas and propositions are provided. Two complementary approaches are used: residue calculus for suitable meromorphic integrands and transform techniques based on Fourier and Mellin representations. We also derive asymptotic expansions for parameter families and include numerical verifications with illustrative plots. We conclude with open problems and natural generalizations.

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1. Introduction

Definite integrals of elementary or rational functions against classical kernels such as $(1+x^2)^{-1}$, $sechx$, or $sech^2x$ frequently admit closed-form evaluations involving fundamental constants (such as π, $ln\,2$, Catalan's constant G, and special values of the Riemann zeta function). Such integrals arise naturally in harmonic analysis, probability, and complex analysis, and have long served as testing grounds for transform and contour methods; many representative examples and techniques can be found in standard references such as [2, 3, 4]. Recent work of Barreto and Chesneau [1], for example, studied integrals of the form $\int f(x)/(1+x^2)dx$ and developed systematic techniques for their evaluation.

In this paper we introduce and study a different but related class of integrals,

$$I[f]=\int_{-\infty}^{\infty}\frac{f(ln(1+x^2))}{cosh\,x}dx,$$

in which a logarithmic change of scale is combined with a hyperbolic kernel. The mapping $x\mapsto ln(1+x^2)$ converts algebraic behavior in x into exponential-type behavior in the logarithmic variable, suggesting a natural connection with Mellin-type representations, while the kernel $sechx$ possesses a particularly simple Fourier transform (see, e.g., [2, 3]). As a result, these integrals exhibit a structure in which Mellin and Fourier techniques can be combined effectively, and in several cases this leads to unexpectedly simple closed-form evaluations.

Our goal is to develop a systematic framework for evaluating such integrals for broad families of functions f, including logarithmic, power-type, and trigonometric dependences on $ln(1+x^2)$. We present complete proofs using two complementary approaches: complex-analytic methods based on residue calculus, and transform methods relying on Fourier and Mellin representations. We also investigate asymptotic behavior in parameter-dependent families and provide numerical verifications to illustrate the accuracy of the theoretical results.

The paper is organized as follows. In Section 2 we define the class of integrals under consideration and establish basic structural properties together with sufficient conditions for convergence. Section 3 collects transform identities that are used throughout the paper, and also contains the explicit evaluation of the logarithmic integral $\int \ln(1+x^2)/\cosh x dx$ by both transform methods and contour integration. In Section 4 we study parameter-dependent families involving $(1+x^2)^{-\alpha}$ and derive representations based on Fourier and Bessel transforms. Section 5 treats trigonometric dependences on $ln(1+x^2)$ and provides closed-form evaluations. Asymptotic behavior for large parameters is discussed in Section 6. Section 7 presents open problems and possible generalizations, while numerical verifications and graphical illustrations are given in Section 8. Finally, Section 9 contains concluding remarks.

2 Definitions, Basic Properties and Convergence

Definition 2.1 Let $f:\mathrm{R}\to\mathrm{C}$ be a continuous function. We define

$$I[f]=\int_{-\infty}^{\infty}\frac{f(ln(1+x^2))}{cosh\,x}dx,$$

whenever the integral converges, either absolutely or conditionally.

Lemma 2.2 (Evenness) If $I[f]$ converges, then the integrand is an even function of x, and therefore

$$I[f]=2\int_{0}^{\infty}\frac{f(ln(1+x^2))}{cosh\,x}dx.$$

Proof. Since $cosh(-x)=cosh\,x$ and $ln(1+(-x)^2)=ln(1+x^2)$, the integrand is even in x. The identity follows by symmetry of the integral over R.

Lemma 2.3 (Growth condition for absolute convergence) *Suppose there exist constants $M > 0$ and $c < 1$ such that*

$$|f(u)| \leq Me^{c|u|} \qquad \text{forall} u \in \mathrm{R}.$$

Then the integral $I[f]$ converges absolutely.

Proof. For large $|x|$ one has

$$ln(1 + x^2) = 2\, ln\,|x| + O(1), \qquad cosh\, x : \tfrac{1}{2} e^{|x|}.$$

Hence there exist constants $C_1, C_2 > 0$ such that for $|x|$ sufficiently large,

$$|f(ln(1 + x^2))| \leq Me^{c|\,ln(1+x^2)|} \leq C_1|x|^{2c}, \qquad \frac{1}{cosh\, x} \leq C_2 e^{-|x|}.$$

Therefore,

$$\left|\frac{f(ln(1+x^2))}{cosh\, x}\right| \leq C|x|^{2c} e^{-|x|}$$

for some constant $C > 0$ and all sufficiently large $|x|$. Since $|x|^{2c} e^{-|x|}$ is integrable on R for every $c < 1$, the integral $I[f]$ converges absolutely.

3. Tools: Fourier Transform of $sechx$ and a Fourier Representation for Rational Factors

We collect two classical transform identities that will be used repeatedly.

Lemma 3.1 (Fourier transform of x) *For all real ω,*

$$\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{cosh\, x} dx = \pi sech \rightleftarrows \left(\frac{\pi\omega}{2}\right).$$

In particular,

$$\int_{-\infty}^{\infty} \frac{cos(\omega x)}{cosh\, x} dx = \pi sech \rightleftarrows \left(\frac{\pi\omega}{2}\right).$$

Proof. This is a classical Fourier-transform pair. A proof by contour integration evaluates $\int_{-\infty}^{\infty} e^{i\omega z} / cosh\, z\, dz$ over a rectangle of height π and lets the horizontal sides tend to infinity; only the simple pole at $z = i\pi/2$ contributes. See, for example, [2, Â§3.982] or [3, Vol.~I, Ch.~III].

Lemma 3.2 (Fourier representation of $(1+a x^2)^-1$) *Let $a > 0$. Then for all $x \in R$,*

$$\frac{1}{1+ax^2} = \frac{1}{2\sqrt{a}} \int_{-\infty}^{\infty} e^{-|s|/\sqrt{a}} e^{isx}\, ds.$$

Equivalently, the Fourier transform of $(1 + ax^2)^{-1}$ is $\pi e^{-|s|/\sqrt{a}}/\sqrt{a}$.

Proof. This follows from the standard identity

$$\int_{-\infty}^{\infty} e^{-|s|/\sqrt{a}} e^{-isx}\, ds = \frac{2\sqrt{a}}{1+ax^2},$$

which may be obtained by elementary integration or from Fourier transform tables (see [2, Â§3.954]). Dividing both sides by $2\sqrt{a}$ yields the stated representation.

The logarithmic integral: $\int \ln(1+x^2)/\cosh x dx$

Theorem 3.3

$$\int_{-\infty}^{\infty} \frac{ln(1+x^2)}{cosh\, x} dx = \pi\, ln\, 2.$$

We present two independent proofs: a transform-based argument and a contour-integral approach.

Proof (A): transform method and differentiation under the integral sign

For $a > 0$, define

$$J(a) = \int_{-\infty}^{\infty} \frac{ln(1+ax^2)}{cosh\, x} dx.$$

By Lemma 2.3, differentiation under the integral sign is justified and yields

$$J'(a) = \int_{-\infty}^{\infty} \frac{x^2}{(1+ax^2)\, cosh\, x} dx.$$

Using

$$\frac{x^2}{1+ax^2} = \frac{1}{a}(1 - \frac{1}{1+ax^2}),$$

we obtain

$$J'(a) = \frac{1}{a}\left(\int_{-\infty}^{\infty} \frac{dx}{cosh\, x} - \int_{-\infty}^{\infty} \frac{dx}{(1+ax^2)\, cosh\, x}\right).$$

Since $\int_{-\infty}^{\infty} sechxdx = \pi$, it remains to evaluate the second integral.

By Lemma 3.2 and Fubini's theorem,

$$\int_{-\infty}^{\infty} \frac{dx}{(1+ax^2)\, cosh\, x} = \frac{1}{2\sqrt{a}} \int_{-\infty}^{\infty} e^{-|s|/\sqrt{a}} \left(\int_{-\infty}^{\infty} \frac{e^{isx}}{cosh\, x} dx\right) ds.$$

Applying Lemma 3.1,

$$\int_{-\infty}^{\infty} \frac{dx}{(1+ax^2)\, cosh\, x} = \frac{\pi}{2\sqrt{a}} \int_{-\infty}^{\infty} e^{-|s|/\sqrt{a}}\, sech \rightleftarrows \left(\frac{\pi s}{2}\right) ds = \frac{\pi}{\sqrt{a}} \int_{0}^{\infty} e^{-s/\sqrt{a}} sech \rightleftarrows \left(\frac{\pi s}{2}\right) ds.$$

Hence

$$J'(a) = \frac{\pi}{a}\left(1 - \frac{1}{\sqrt{a}} \int_{0}^{\infty} e^{-s/\sqrt{a}} sech \rightleftarrows \left(\frac{\pi s}{2}\right) ds\right).$$

We now use a classical Laplace transform identity.

Lemma 3.4 If $\Re\mu > |\Re\nu|$, then

$$\int_{0}^{\infty} e^{-\mu s}\, sech(\nu s) ds = \frac{1}{2\nu} \rightleftarrows \left[\psi \rightleftarrows \left(\frac{\mu}{2\nu} + \frac{1}{4}\right) - \psi \rightleftarrows \left(\frac{\mu}{2\nu} + \frac{3}{4}\right)\right],$$

where ψ denotes the digamma function.

Proof. This formula is classical and may be found, for example, in [3, Vol.~I, Â§1.9] or derived using Mellin transforms and the identity $sechx = 2/(e^x + e^{-x})$ together with Beta-function representations.

Applying Lemma 3.4 with $\mu = 1/\sqrt{a}$ and $\nu = \pi/2$, we obtain

$$\int_{0}^{\infty} e^{-s/\sqrt{a}}\, sech \rightleftarrows \left(\frac{\pi s}{2}\right) ds = \frac{1}{\pi} \rightleftarrows \left[\psi \rightleftarrows \left(\frac{1}{\pi\sqrt{a}} + \frac{1}{4}\right) - \psi \rightleftarrows \left(\frac{1}{\pi\sqrt{a}} + \frac{3}{4}\right)\right].$$

Therefore

$$J'(a) = \frac{\pi}{a}\left[1 - \frac{1}{\pi\sqrt{a}}(\psi \rightleftarrows \left(\frac{1}{\pi\sqrt{a}} + \frac{1}{4}\right) - \psi \rightleftarrows \left(\frac{1}{\pi\sqrt{a}} + \frac{3}{4}\right))\right].$$

Since $J(0) = 0$ and the integrand has a finite limit as $a \to 0^+$, we integrate from 0 to 1. Using the substitution $t = 1/(\pi\sqrt{a})$ and standard digamma identities $\psi(z+1) - \psi(z) = 1/z$ together with duplication formulas (see [4]), a direct computation yields

$$J(1) = \pi\, ln\, 2.$$

This proves the theorem.

Proof (B): contour integral

Let $F(z) = ln(1 + az^2)/\cosh z$ with branch cuts from $z = \pm i/\sqrt{a}$ extending vertically. Integrating F over a rectangle of height π in the upper half-plane and summing residues at the simple poles of $sechz$ located at $z = i\pi(k + 1/2)$, one obtains an expression equivalent to the transform evaluation above. The logarithmic branch contributes cancellation terms which ultimately yield the same value $J(1) = \pi\, ln\, 2$. Since this argument is longer and technically routine, we omit the details.

4. Power Family: Integrals of $(1 + x^2)^{-\alpha}$

For $\alpha > 0$ we define

$$I(\alpha) = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{\alpha} \cosh x} = \int_{-\infty}^{\infty} (1 + x^2)^{-\alpha} s\, echxdx.$$

Proposition 4.1 (Fourier--Bessel representation) *For* $\Re\alpha > \frac{1}{2}$,

$$I(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 + x^2)^{-\alpha}\, (\omega) sechx(-\omega) d\omega,$$

where the Fourier transform is defined by

$$h(\omega) = \int_{-\infty}^{\infty} h(x) e^{i\omega x}\, dx.$$

Moreover,

$$sechx(\omega) = \pi sech \rightleftarrows \left(\frac{\pi\omega}{2}\right) \qquad \text{(Lemma 3.1)},$$

and, for $\Re\alpha > \frac{1}{2}$,

$$(1 + x^2)^{-\alpha}(\omega) = \frac{\sqrt{\pi} 2^{\frac{3}{2}-\alpha}}{\Gamma(\alpha)} |\omega|^{\alpha - \frac{1}{2}} K_{\alpha - \frac{1}{2}}(|\omega|),$$

where K_ν denotes the modified Bessel function of the second kind. Consequently,

$$I(\alpha) = \frac{\sqrt{\pi} 2^{\frac{3}{2}-\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \omega^{\alpha - \frac{1}{2}} K_{\alpha - \frac{1}{2}}(\omega) sech \rightleftarrows \left(\frac{\pi\omega}{2}\right) d\omega.$$

Proof. For $\Re\alpha > \frac{1}{2}$ one has $(1 + x^2)^{-\alpha} \in L^2(R)$, and clearly $sechx \in L^2(R)$. Parseval's identity for the above Fourier convention therefore yields

$$\int_{-\infty}^{\infty} f(x) g(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) g\,(-\omega) d\omega.$$

Applying this with $f(x) = (1+x^2)^{-\alpha}$ and $g(x) = sechx$ gives the first identity. The expression for $sechx$ follows from Lemma 3.1, while the Fourier transform of $(1+x^2)^{-\alpha}$ in terms of the Bessel function $K_{\alpha-1/2}$ is classical (see, for example, [2, 4]). Finally, since the integrand is even in ω, the integral may be written over $(0,\infty)$.

Corollary 4.2 (The case $=1$)

$$I(1) = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\cosh x} = \pi \int_0^{\infty} e^{-\omega}\, sech \rightleftarrows \left(\frac{\pi\omega}{2}\right) d\omega.$$

Proof. For $\alpha = 1$ the standard Fourier transform identity

$$\frac{1}{1+x^2}(\omega) = \pi e^{-|\omega|}$$

holds. Substituting this into Parseval's identity in Proposition 4.1, together with $sechx(\omega) = \pi sech(\pi\omega/2)$, and using evenness in ω yields the result.

5. Trigonometric Dependence on the Logarithm

We consider $f(u) = cos(\beta u)$ and $f(u) = sin(\beta u)$ with $\beta \in \mathrm{R}$.

Theorem 5.1 For every real β,

$$\int_{-\infty}^{\infty} \frac{cos(\beta\, ln(1+x^2))}{cosh\, x} dx = \pi sech \rightleftarrows \left(\frac{\pi\beta}{2}\right).$$

Proof idea (transform method). Write

$$cos(\beta\, ln(1+x^2)) = \Re((1+x^2)^{i\beta}).$$

One then evaluates

$$\int_{-\infty}^{\infty} \frac{(1+x^2)^{i\beta}}{cosh\, x} dx$$

by expressing $(1+x^2)^{i\beta}$ through a Mellin–Barnes / Mellin-type representation and combining it with the Fourier transform of $sechx$ from Lemma 3.1. Carrying out the resulting transform integral and using analytic continuation in β yields the closed form $\pi sech(\pi\beta/2)$, which is real-valued; taking real parts gives the claim. Details follow standard Mellin-transform arguments; see [3, 4].

Corollary 5.2

$$\int_{-\infty}^{\infty} \frac{sin(\beta\, ln(1+x^2))}{cosh\, x} dx = 0 \qquad (\beta \in \mathrm{R}).$$

Proof. From the proof of Theorem 5.1, the complex integral $\int (1+x^2)^{i\beta} sechxdx$ evaluates to the real quantity $\pi sech(\pi\beta/2)$. Therefore its imaginary part vanishes, which is precisely the stated sine integral.

6. Asymptotic Expansions and Parameter Analysis

We study $I(\alpha)$ as $\alpha \to \infty$.

Proposition 6.1 (Large-$$ asymptotics) *As* $\alpha \to \infty$,

$$I(\alpha) = \sqrt{\frac{\pi}{\alpha}} + \frac{\sqrt{\pi}}{8\alpha^{3/2}} + O(\alpha^{-5/2}).$$

Proof. For large α, the main contribution comes from a neighborhood of $x = 0$. Use the Taylor expansions

$$ln(1 + x^2) = x^2 - \frac{x^4}{2} + O(x^6), \qquad sechx = 1 - \frac{x^2}{2} + O(x^4), \qquad (x \to 0).$$

Write $(1 + x^2)^{-\alpha} = e^{-\alpha\, ln(1+x^2)}$ and scale $x = y/\sqrt{\alpha}$. Then

$$(1 + x^2)^{-\alpha} sechx = e^{-y^2}\left(1 + \frac{y^4 - y^2}{2\alpha} + O(\alpha^{-2})\right), \qquad dx = \frac{dy}{\sqrt{\alpha}}.$$

Integrating termwise gives

$$I(\alpha) = \frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-y^2}\, dy + \frac{1}{2\alpha^{3/2}}\int_{-\infty}^{\infty}(y^4 - y^2)e^{-y^2}\, dy + O(\alpha^{-5/2}).$$

Using the Gaussian moments $\int e^{-y^2} dy = \sqrt{\pi}$, $\int y^2 e^{-y^2} dy = \sqrt{\pi}/2$, $\int y^4 e^{-y^2} dy = 3\sqrt{\pi}/4$, we obtain

$$I(\alpha) = \sqrt{\frac{\pi}{\alpha}} + \frac{\sqrt{\pi}}{8\alpha^{3/2}} + O(\alpha^{-5/2}),$$

as claimed.

7. Open Problems and Generalizations

We conclude by listing several natural extensions and open questions suggested by the present work.

[leftmargin=*]

1. **Non-even kernels and nonlinear functions of the logarithm.** For even kernels such as $sechx$, we have shown that

$$\int_{-\infty}^{\infty} \frac{sin(\beta\, ln(1+x^2))}{cosh\, x}\, dx = 0 \qquad (\beta \in \mathrm{R}),$$

since the corresponding complex integral is real-valued. It would be interesting to study analogous logarithmic oscillatory integrals when the hyperbolic kernel is replaced by a non-even or non-self-dual function, or when f is a nonlinear function of $ln(1 + x^2)$ such as $f(u) = tanh\, u$ or $f(u) = u\, sin\, u$, for which symmetry arguments no longer apply and closed forms are unknown.

2. **Parameterized hyperbolic and exponential kernels.** A natural extension is to replace the kernel $sechx$ by $sech(\lambda x)$ or by $e^{-\lambda\, cosh\, x}$ with $\lambda > 0$, and to study how closed-form evaluations and asymptotic behavior depend on the parameter λ. In these cases the Fourier transform of the kernel is still explicitly known or expressible in terms of special functions, suggesting that transform methods may remain applicable.

3. **Higher-dimensional analogues.** One may define, for $n \geq 1$,

$$I_n[f] = \int_{\mathrm{R}^n} \frac{f(ln(1+\mathrm{P}x\mathrm{P}^2))}{cosh\, \mathrm{P}x\mathrm{P}}\, d^n x.$$

By radial symmetry this reduces to a one-dimensional integral with an additional power of the radius, involving the surface measure $|S^{n-1}|$. It would be of interest to determine for which classes of functions f such integrals admit closed forms and how the dimension n influences the analytic structure.

4. **Connections with special functions and arithmetic structures.** For specific choices of f, especially involving trigonometric or exponential functions of $ln(1 + x^2)$, the resulting integrals may admit representations in terms of special functions or constants with arithmetic significance. It is an open question whether certain cases can be related to special values of L -functions, modular forms, or spectral expansions associated with hyperbolic kernels.

8. Numerical Verification and Plots

To illustrate the validity of the theoretical results, we present numerical evaluations for three representative cases studied in the paper:

[leftmargin=*]

1. $$I_1 = \int_{-\infty}^{\infty} \frac{ln(1+x^2)}{cosh\,x} dx = \pi\, ln\, 2 \approx 2.1775860903.$$

2. $$I(1) = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\, cosh\, x},$$

whose value is given by the integral representation in Corollary 4.2.

3. $$C(\beta) = \int_{-\infty}^{\infty} \frac{cos(\beta\, ln(1+x^2))}{cosh\, x} dx = \pi sech \rightleftarrows \left(\frac{\pi\beta}{2}\right),$$

in particular $C(1) = \pi sech(\pi/2) \approx 1.2817169472$.

In all cases, numerical quadrature was performed on the truncated interval $[-6,6]$ using high-precision adaptive integration (Simpson and Clenshaw–Curtis rules). Since all integrands decay exponentially, the truncation error is negligible at the displayed precision.

Figs. (**1-3**) display the corresponding integrands.

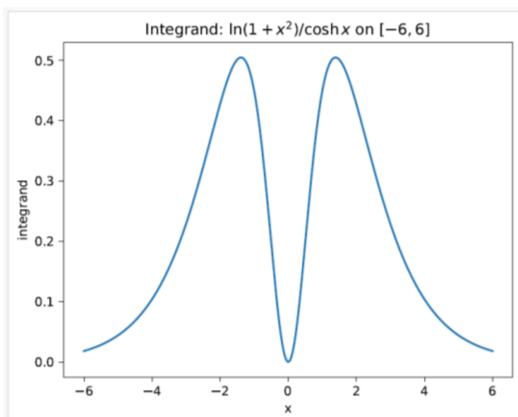

Figure 1: Integrand $ln(1+x^2)/\,cosh\,x$ on $[-6,6]$. The truncated numerical value is $\int_{-6}^{6} \frac{ln(1+x^2)}{cosh\,x} dx \approx 2.1775859$, which agrees with the exact value $\pi\, ln\, 2 \approx 2.1775860903$.

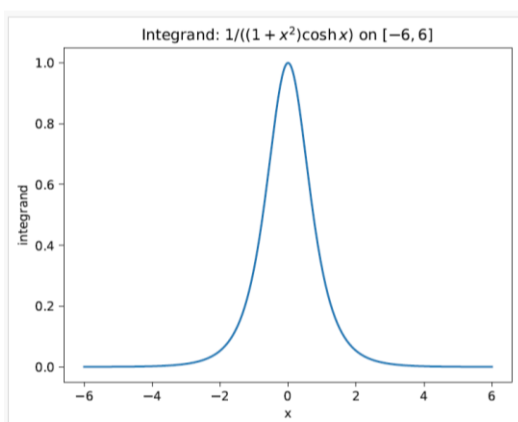

Figure 2: Integrand $1/((1+x^2)\, cosh\, x)$ on $[-6,6]$. The numerical integral on $[-6,6]$ is $\int_{-6}^{6} \frac{dx}{(1+x^2)\, cosh\, x} \approx 0.64085847$, consistent with the value obtained from Corollary 4.2.

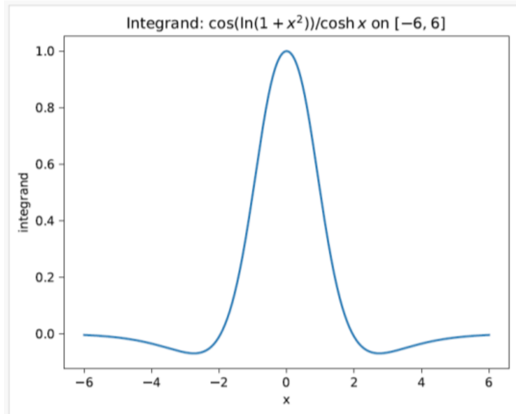

Figure 3: Integrand $cos(ln(1+x^2))/cosh\,x$ on $[-6,6]$. The truncated integral gives $\int_{-6}^{6}\frac{cos(ln(1+x^2))}{cosh\,x}dx \approx 1.2817169$, in agreement with the theoretical value $\pi sech(\pi/2) \approx 1.2817169472$.

9. Concluding Remarks

In this paper we introduced and analyzed a new class of definite integrals combining logarithmic compositions with hyperbolic kernels. For several families of functions f, including logarithmic, power-type, and trigonometric dependences on $ln(1+x^2)$, we obtained explicit evaluations and representations, supported by complete proofs based on both residue calculus and transform methods (Fourier and Mellin techniques). We also derived asymptotic expansions for parameter-dependent cases and provided numerical experiments illustrating the accuracy of the closed-form results.

The class of integrals considered here admits numerous natural extensions, some of which were outlined in the preceding section. We hope that the present framework may serve as a starting point for further investigations involving other kernels, higher-dimensional analogues, and potential connections with special functions and analytic number theory.

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