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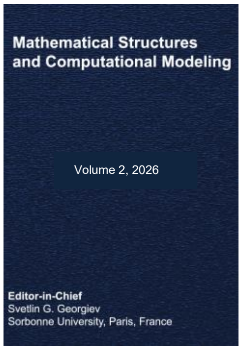

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On Mapping Properties of Certain Generalized Integral Operators

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ABSTRACT

In this article, we investigate mapping properties of certain generalized integral operators for the generalized classes of functions of bounded boundary and bounded radius rotations.

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1. Introduction and Preliminaries

Let A denote the class of all functions $f(z)$ which are analytic and locally univalent in the open unit disk $D = \{z: |z| < 1\}$ and satisfy the normalization condition $f(0) = f^{'}(0) - 1 = 0$.

Let $P_m(c, \delta)$ denote the class of all functions $p(z)$ analytic in D with $p(0) = 1$ and satisfy

$$\int_0^{2\pi} \left| \frac{R\left(1 + \frac{1}{c}[p(z) - 1]\right) - \delta}{1 - \delta} \right| d\theta \leq m\pi; \quad (z = re^{i\theta})$$

where $c \in \mathbb{C} - \{0\}$, $0 \leq \delta < 1$ and $m \geq 2$. This class was introduced by Noor *et al*. [13].

For special values of the involved parameters we get the following.

1. For $m = 2$, $P_2(c, \delta) = P(c, \delta)$.
2. For $c = 1$, $P_m(1, \delta) = P_m(\delta)$; studied in [15].
3. For $m = 2, c = 1$, we have $P_2(1, \delta) = P_m$; introduced in [17].
4. For $m = 2, \delta = 0$, we have $P_2(c, 0) = P(c)$; discussed in [11].
5. For $c = 1, m = 2$, we receive $P_2(1, \delta) = P(\delta)$ and for $c = 1, m = 2, \delta = 0$, we receive $P_2(1,0) = P$; see [4].

In relation to $P_m(c, \delta)$, we now define the class $P_m(c, \delta, \tau)$ of functions $p(z)$ with $p(0) = 1$ that satisfy

$$\int_0^{2\pi} \left| \frac{R\left\{e^{i\tau}\left(1 + \frac{1}{c}[p(z) - 1]\right) - \delta \cos \tau\right\}}{1 - \delta} \right| d\theta \leq m\pi \cos \tau; \quad (z = re^{i\theta})$$

where $c \in \mathbb{C} - \{0\}$, $0 \leq \delta < 1$, $m \geq 2$ and τ is real with $|\tau| < \frac{\pi}{2}$.

For $\delta = 0$, $P_m(c, \delta, \tau)$ immediately gives the class $P_m(c, \tau)$ defined by Ahuja *et al*. [1]; whereas for $\tau = 0$ we have class $P_m(c, \delta)$ as defined above. For $c = 1$, we have $P_m(1, \delta, \tau) = P_m(\delta, \tau)$; see [12]. On similar pattern by varying the involved parameters we get a number of known classes as observed above.

A function $f \in A$ is said to be in the class $V_m(c, \delta)$ of bounded boundary rotation of complex order, if $\left(1 + \frac{zf^{''}(z)}{f^{'}(z)}\right) = p(z) \in P_m(c, \delta)$. We note that $f \in V_m(c, \delta)$ iff

$$\left(1 + \frac{1}{c}\left[\frac{zf^{''}(z)}{f^{'}(z)}\right]\right) \in P_m(\delta), \quad (z \in D).$$

We say that a function $f \in A$ is in the class $R_m(c, \delta)$ of bounded radius rotation of complex order, if $\frac{zf^{'}(z)}{f(z)} = p(z) \in P_m(c, \delta)$. It can be easily verified that

$$f \in V_m(c, \delta) \Leftrightarrow zf^{'} \in R_m(c, \delta).$$

A function $f \in A$ is said to be in the class $S_\tau^*(c)$ of spiral-like of complex order, iff (see [6]):

$$e^{i\tau}\left(1 + \frac{1}{c}\left[\frac{zf^{'}(z)}{f(z)} - 1\right]\right) \in P, \quad (z \in D).$$

In other words, $f \in S_\tau^*(c)$ iff $\frac{zf^{'}(z)}{f(z)} = p(z) \in P(c, \tau)$. For $c = 1$, we retrieve the class S_τ^* of spiral-like functions introduced by Spacek [20], and for $\tau = 0$, we have the well known class of starlike functions of complex order S_c^* [5]. Further generalization of starlike and spirallike functions can be found in [8, 7, 22] and the reference therein.

We say that f is in the class $C_\tau(c)$, if $zf' \in S_\tau^*(c)$. For $c = 1$, we retrieve the class C_τ of Robertson functions [19] and for $c = 1, \tau = 0$, we have the well known class of convex functions C [4].

We now define the following classes.

$$R_m(c,\delta,\tau) = \left\{f \in A: \frac{zf'(z)}{f(z)} \in P_m(c,\delta,\tau)\right\}, \quad (z \in D),$$

$$V_m(c,\delta,\tau) = \left\{f \in A: 1 + \frac{zf''(z)}{f'(z)} \in P_m(c,\delta,\tau)\right\}, \quad (z \in D).$$

On letting $\tau = 0$ we retrieve the classes $R_m(c,\delta)$ and $V_m(c,\delta)$ respectively as defined above and studied by Noor *et al*. [13] and for $c = 1$, see [12]. Also, we note that $R_2(c,0,\tau) = S_\tau^*(c)$ and $V_2(c,0,\tau) = C_\tau(c)$. For the well-known classes of bounded radius $R_m := R_m(1,0,0)$ and bounded boundary $V_m := V_m(1,0,0)$ rotations, see [14, 21]. For some recent investigation see [18].

Now we define the integral operators $F_{\alpha_i,\beta_i}(z)$ and $H_{\alpha_i,\beta_i}(z)$ as follows:

$$F_{\alpha_i,\beta_i}(z) = \int_0^z \prod_{i=1}^n \left[\frac{f_i(t)}{t}\right]^{\alpha_i} \left[\frac{g_i(t)}{t}\right]^{\beta_i} dt, \tag{1}$$

and

$$H_{\alpha_i,\beta_i}(z) = \int_0^z \prod_{i=1}^n [f_i'(t)]^{\alpha_i}[g_i'(t)]^{\beta_i} dt, \tag{2}$$

where each $\alpha_i, \beta_i > 0$ and $f_i, g_i \in A$ for $1 \le i \le n$.

Remark 1 These operators generalize many known operators. For instance, the integral operator in (1) for $\beta_i = 0$ $(1 \le i \le n)$ we have $F_n(z) = \int_0^z \prod_{i=1}^n [f_i(t)/t]^{\alpha_i} dt$, which was introduced by Breaz and Breaz [2]; whereas the integral operator in (2) for $\beta_i = 0$ $(1 \le i \le n)$ reduces to $G_n(z) = \int_0^z \prod_{i=1}^n [f_i'(t)]^{\alpha_i} dt$, which was considered by Breaz, Owa and Breaz [3]. Similarly, on letting $n = 1$, $\alpha_1 = \alpha$, $\beta_1 = \beta$ in (2) we get the integral operator $F_{\alpha,\beta}(z) = \int_0^z [f'(t)]^\alpha \, [g'(t)]^\beta dt$ studied in [13]. Also, on letting $n = 1$, $\alpha_1 = \alpha$, $\beta_1 = 0$ in (1) and (2), we have the integral operators $F(z) = \int_0^z [f(t)\,/t]^\alpha dt$ and $H(z) = \int_0^z [f'(t)]^\alpha \, dt$ discussed in [10] and [9, 16] respectively.

The study of integral operators in geometric function theory has a rich history, dating back to the classical Alexander and Libera operators [4, 23]. These operators serve as fundamental tools for generating new analytic functions with prescribed geometric properties from known ones. The generalized operators $F_{\alpha_i,\beta_i}(z)$ and $H_{\alpha_i,\beta_i}(z)$ introduced in this work are motivated essentially by the mapping properties of the operators $F_n(z)$, $G_n(z)$ and $F_{\alpha,\beta}(z)$ over the classes of bounded boundary and boundary radius rotations $V_m(c,\delta)$ and $R_m(c,\delta)$ (see Noor *et al*. [13]) and $V_m(c,\tau)$ and $R_m(c,\tau)$ (see Ahuja *et al*. [1]). The bounded boundary and bounded radius rotation conditions, originally introduced by Paatero [14] and Tammi [21] respectively, has deep connections with the theory of univalent functions and quasiconformal mappings. Functions of bounded boundary rotation arise naturally in the study of conformal mappings onto convex and close-to-convex domains. By extending these concepts to include complex order and spiral angle parameters, we gain insights into more general mapping properties that may have applications in the theory of differential equations and special functions, numerical conformal mapping and computational geometry, and in the study of harmonic mappings and minimal surfaces.

From now, we shall assume that $m \ge 2$, $c \in \mathbb{C} - \{0\}$, $\delta \in [0,1)$ and $z \in D$, unless otherwise stated.

2. Main Results

Now, we prove the following main results for the classes $V_m(c,\delta)$ and $V_m(c,\delta,\tau)$.

Theorem 1 Let $f_i, g_i \in R_m(c, \delta_i)$ $(1 \le i \le n)$ with $0 \le \delta_i < 1$ and let each $\alpha_i, \beta_i > 0$ $(1 \le i \le n)$. If $\sum_{i=1}^{n} (\alpha_i + \beta_i) \neq 0$ and

$$0 \le \left[1 - \sum_{i=1}^{n} (\alpha_i + \beta_i)\right] + \sum_{i=1}^{n} (\alpha_i + \beta_i)\delta_i < 1,$$

then the integral operator given by (1) is in $V_m(c, \gamma)$ where $\gamma = [1 - \sum_{i=1}^{n} (\alpha_i + \beta_i)] + \sum_{i=1}^{n} (\alpha_i + \beta_i)\delta_i$.

Proof. Logarithmic differentiation of (1) and subsequent calculation gives

$$1 + \frac{1}{c}\left[\frac{zF''_{\alpha_i,\beta_i}(z)}{F'_{\alpha_i,\beta_i}(z)}\right] = 1 - \sum_{i=1}^{n} (\alpha_i + \beta_i) + \sum_{i=1}^{n} \alpha_i \left(1 + \frac{1}{c}\left[\frac{zf_i'(z)}{f_i(z)} - 1\right]\right) + \sum_{i=1}^{n} \beta_i \left(1 + \frac{1}{c}\left[\frac{zg_i'(z)}{g_i(z)} - 1\right]\right).$$

Now letting $1 + \frac{1}{c}\left[\frac{zf_i'(z)}{f_i(z)} - 1\right] = h_i(z)$ and $1 + \frac{1}{c}\left[\frac{zg_i'(z)}{g_i(z)} - 1\right] = k_i(z)$ $(1 \le i \le n)$ for some $h_i, k_i \in P_m(\delta_i)$, we get

$$1 + \frac{1}{c}\left[\frac{zF''_{\alpha_i,\beta_i}(z)}{F'_{\alpha_i,\beta_i}(z)}\right] = \left[1 - \sum_{i=1}^{n} (\alpha_i + \beta_i)\right] + \sum_{i=1}^{n} \alpha_i h_i(z) + \sum_{i=1}^{n} \beta_i k_i(z).$$

Since $P_m(\delta_i)$ is convex set [13], and $\sum_{i=1}^{n}(\alpha_i + \beta_i) \neq 0$; thus, we have

$$1 + \frac{1}{c}\left[\frac{zF''_{\alpha_i,\beta_i}(z)}{F'_{\alpha_i,\beta_i}(z)}\right] = \left[1 - \sum_{i=1}^{n} (\alpha_i + \beta_i)\right] + \sum_{i=1}^{n} (\alpha_i + \beta_i)\left[\frac{\sum_{i=1}^{n} \alpha_i}{\sum_{i=1}^{n}(\alpha_i + \beta_i)} h_i(z) + \frac{\sum_{i=1}^{n} \beta_i}{\sum_{i=1}^{n}(\alpha_i + \beta_i)} k_i(z)\right].$$

Or equivalently, for some $p_i \in P_m(\delta_i)$, set $p_i(z) = \frac{\sum_{i=1}^{n} \alpha_i}{\sum_{i=1}^{n}(\alpha_i + \beta_i)} h_i(z) + \frac{\sum_{i=1}^{n} \beta_i}{\sum_{i=1}^{n}(\alpha_i + \beta_i)} k_i(z)$ $(1 \le i \le n)$ so that

$$1 + \frac{1}{c}\left[\frac{zF''_{\alpha_i,\beta_i}(z)}{F'_{\alpha_i,\beta_i}(z)}\right] = \left[1 - \sum_{i=1}^{n} (\alpha_i + \beta_i)\right] + \sum_{i=1}^{n} (\alpha_i + \beta_i) p_i(z) = \varphi(z),$$

where

$$\varphi \in P_m(\gamma); \quad \gamma = \left[1 - \sum_{i=1}^{n} (\alpha_i + \beta_i)\right] + \sum_{i=1}^{n} (\alpha_i + \beta_i)\delta_i.$$

Hence $F_{\alpha_i,\beta_i}(z) \in V_m(c, \gamma)$ which completes the proof.

Theorem 2 Let $f_i, g_i \in V_m(c, \delta_i)$ $(1 \le i \le n)$ with $0 \le \delta_i < 1$ and let each $\alpha_i, \beta_i > 0$ $(1 \le i \le n)$. If $\sum_{i=1}^{n} (\alpha_i + \beta_i) \neq 0$ and

$$0 \le \left[1 - \sum_{i=1}^{n} (\alpha_i + \beta_i)\right] + \sum_{i=1}^{n} (\alpha_i + \beta_i)\delta_i < 1,$$

then the integral operator given by (2) is in $V_m(c, \gamma)$ where $\gamma = [1 - \sum_{i=1}^{n}(\alpha_i + \beta_i)] + \sum_{i=1}^{n}(\alpha_i + \beta_i)\delta_i$.

Proof. Logarithmic differentiation of (2) and some easy calculation gives

$$1 + \frac{1}{c}\left[\frac{zH''_{\alpha_i,\beta_i}(z)}{H'_{\alpha_i,\beta_i}(z)}\right] = \left[1 - \sum_{i=1}^{n} (\alpha_i + \beta_i)\right] + \sum_{i=1}^{n} \alpha_i \left[1 + \frac{1}{c}\left(\frac{zf_i''(z)}{f_i'(z)}\right)\right] + \sum_{i=1}^{n} \beta_i \left[1 + \frac{1}{c}\left(\frac{zg_i''(z)}{g_i'(z)}\right)\right].$$

Now setting $1 + \frac{1}{c}\left[\frac{zf_i''(z)}{f_i'(z)}\right] = h_i(z)$ and $1 + \frac{1}{c}\left[\frac{zg_i''(z)}{g_i'(z)}\right] = k_i(z)$ $(1 \le i \le n)$ for some $h_i, k_i \in P_m(\delta_i)$, we have

$$1+\frac{1}{c}\left[\frac{zH''_{\alpha_i,\beta_i}(z)}{H'_{\alpha_i,\beta_i}(z)}\right]=\left[1-\sum_{i=1}^{n}(\alpha_i+\beta_i)\right]+\sum_{i=1}^{n}\alpha_i h_i(z)+\sum_{i=1}^{n}\beta_i k_i(z)$$

$$=\sum_{i=1}^{n}(\alpha_i+\beta_i)\left[\frac{\sum_{i=1}^{n}\alpha_i}{\sum_{i=1}^{n}(\alpha_i+\beta_i)}h_i(z)+\frac{\sum_{i=1}^{n}\beta_i}{\sum_{i=1}^{n}(\alpha_i+\beta_i)}k_i(z)\right]$$

$$+\left[1-\sum_{i=1}^{n}(\alpha_i+\beta_i)\right]$$

$$=\left[1-\sum_{i=1}^{n}(\alpha_i+\beta_i)\right]+\sum_{i=1}^{n}(\alpha_i+\beta_i)\,p_i(z);\quad \text{forsome} p_i\in P_m(\delta_i)$$

$$=\varphi(z);\quad \varphi\in P_m(\gamma);\quad \gamma=\left[1-\sum_{i=1}^{n}(\alpha_i+\beta_i)\right]+\sum_{i=1}^{n}(\alpha_i+\beta_i)\,\delta_i.$$

Hence $H_{\alpha_i,\beta_i}(z)\in V_m(c,\gamma)$ which completes the proof.

Theorem 3 Let $f_i,g_i\in R_m(c,\delta_i,\tau)$ $(1\le i\le n)$ with $0\le\delta_i<1$ and let each $\alpha_i,\beta_i>0$ $(1\le i\le n)$. If $\sum_{i=1}^{n}(\alpha_i+\beta_i)\neq 0$ and

$$0\le\left[1-\sum_{i=1}^{n}(\alpha_i+\beta_i)\right]+\sum_{i=1}^{n}(\alpha_i+\beta_i)\delta_i<1,$$

then the integral operator given by (1) is in $V_m(c,\gamma,\tau)$, where $\gamma=\left[1-\sum_{i=1}^{n}(\alpha_i+\beta_i)\right]+\sum_{i=1}^{n}(\alpha_i+\beta_i)\,\delta_i$.

Proof. As in Theorem 1, differentiating (1) gives

$$e^{i\tau}\left(1+\frac{1}{c}\left[\frac{zF''_{\alpha_i,\beta_i}(z)}{F'_{\alpha_i,\beta_i}(z)}\right]\right)=e^{i\tau}\left[1-\sum_{i=1}^{n}(\alpha_i+\beta_i)\right]$$

$$+e^{i\tau}\left(\sum_{i=1}^{n}\alpha_i\left[1+\frac{1}{c}\left(\frac{zf'_i(z)}{f_i(z)}-1\right)\right]+\sum_{i=1}^{n}\beta_i\left[1+\frac{1}{c}\left(\frac{zg'_i(z)}{g_i(z)}-1\right)\right]\right).$$

Following Theorem 1 and convexity of $P_m(\delta)$, we can easily get

$$e^{i\tau}\left(1+\frac{1}{c}\left[\frac{zF''_{\alpha_i,\beta_i}(z)}{F'_{\alpha_i,\beta_i}(z)}\right]\right)=e^{i\tau}\left[1-\sum_{i=1}^{n}(\alpha_i+\beta_i)\right]+\sum_{i=1}^{n}(\alpha_i+\beta_i)\,e^{i\tau}p_i(z);\quad \text{forsome} p_i\in P_m(\delta_i)$$

$$=e^{i\tau}\left[1-\sum_{i=1}^{n}(\alpha_i+\beta_i)\right]+\sum_{i=1}^{n}(\alpha_i+\beta_i)\,q_i(z);\quad \text{forsome} q_i\in P_m(\delta_i,\tau)$$

$$=p(z);\quad p\in P_m(\gamma,\tau);\quad \gamma=\left[1-\sum_{i=1}^{n}(\alpha_i+\beta_i)\right]+\sum_{i=1}^{n}(\alpha_i+\beta_i)\delta_i.$$

Hence $F_{\alpha_i,\beta_i}(z)\in V_m(c,\gamma,\tau)$ which completes the proof.

Theorem 4 Let $f_i,g_i\in V_m(c,\delta_i,\tau)$ $(1\le i\le n)$ with $0\le\delta_i<1$ and let each $\alpha_i,\beta_i>0$ $(1\le i\le n)$. If $\sum_{i=1}^{n}(\alpha_i+\beta_i)\neq 0$ and

$$0\le\left[1-\sum_{i=1}^{n}(\alpha_i+\beta_i)\right]+\sum_{i=1}^{n}(\alpha_i+\beta_i)\delta_i<1,$$

then the integral operator given by (2) is in $V_m(c,\gamma,\tau)$, where $\gamma=\left[1-\sum_{i=1}^{n}(\alpha_i+\beta_i)\right]+\sum_{i=1}^{n}(\alpha_i+\beta_i)\delta_i$.

Proof. Exploiting the same techniques as employed in Theorem 1-3, the proof can be accomplished.

Remark 2 For $\tau=0$, Theorems 1-4 produce the corresponding results in [13, Theorems 4.3 and 4.4]; while, for $\delta=0$ they produce the results in [1, Theorems 4.1 and 4.4]. Other appropriate choices of the involved parameters in Theorems 1-4 can lead to some new and known results for the operators discussed in Remark 1.

3. Illustrative Examples and Applications

To demonstrate the practical significance of our results and make them more accessible, we present several concrete examples and discuss potential applications.

3.1. Example 1: Construction of Convex Functions (Trivial Case)

Consider the simple case where $n = 1$, $\alpha_1 = \frac{1}{2}$, $\beta_1 = \frac{1}{2}$, and let $f_1(z) = g_1(z) = z$. Then both f_1 and g_1 belong to $R_m(c, 0)$ since $\frac{zf_1'(z)}{f_1(z)} = 1 \in P_m(0)$. Then $\sum_{i=1}^{n}(\alpha_i + \beta_i) = 1$, and condition (2.1) gives:

$$0 \leq [1 - 1] + 1 \cdot 0 = 0 < 1,$$

which is satisfied with $\gamma = 0$. By Theorem 2.1, the integral operator becomes:

$$F_{\frac{1}{2},\frac{1}{2}}(z) = \int_0^z \left[\frac{t}{t}\right]^{\frac{1}{2}} \left[\frac{t}{t}\right]^{\frac{1}{2}} dt = \int_0^z dt = z.$$

Thus $F(z) = z \in V_m(c, 0)$, which is indeed the identity mapping — a trivial convex function.

3.2. Example 2: Non-trivial Construction Using Starlike Functions

Let $n = 1$, $\alpha_1 = \frac{1}{3}$, $\beta_1 = \frac{1}{3}$, and choose:

$$f_1(z) = \frac{z}{1 - z}, \quad g_1(z) = \frac{z}{1 - z}.$$

Both functions are the well-known Koebe function, which are starlike in the unit disk. For f_1, we have:

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1}{1 - z}.$$

It is known that $\Re\left(\frac{1}{1-z}\right) > \frac{1}{2}$ for $|z| < 1$, so $f_1 \in R_m(c, \delta_1)$ with $\delta_1 = \frac{1}{2}$. Similarly, $g_1 \in R_m(c, \delta_2)$ with $\delta_2 = \frac{1}{2}$. Then $\sum_{i=1}^{n}(\alpha_i + \beta_i) = \frac{2}{3}$, and:

$$\gamma = \left[1 - \frac{2}{3}\right] + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

The integral operator F gives:

$$F_{\frac{1}{3},\frac{1}{3}}(z) = \int_0^z \left[\frac{1}{1 - t}\right]^{\frac{1}{3}} \left[\frac{1}{1 - t}\right]^{\frac{1}{3}} dt = \int_0^z (1 - t)^{-\frac{2}{3}} dt.$$

Evaluating the integral:

$$F_{\frac{1}{3},\frac{1}{3}}(z) = \left[\frac{(1 - t)^{1-\frac{2}{3}}}{1 - \frac{2}{3}}\right]_{t=0}^{t=z} = 3\left[1 - (1 - z)^{\frac{1}{3}}\right].$$

By Theorem 2.1, $F(z) \in V_m(c, \frac{2}{3})$, meaning it has bounded boundary rotation. This function maps the unit disk onto a domain with a corner at $w = 3$ corresponding to $z = 1$.

3.3. Example 3a: Spiral-like Case (Using Operator $\boldsymbol{F}$)

Let $n = 2$, with $\alpha_1 = \alpha_2 = \frac{1}{4}$, $\beta_1 = \beta_2 = \frac{1}{4}$, so $\sum_{i=1}^{2}(\alpha_i + \beta_i) = 1$. Choose the following functions:

$$f_1(z) = z \quad \text{(identity)},$$

$$f_2(z) = \frac{z}{(1-z)^2} \quad \text{(Koebefunction)},$$

$$g_1(z) = z \quad \text{(identity)},$$

$$g_2(z) = \frac{z}{1-z} \quad \text{(starlikefunction)}.$$

For these functions, we compute the required quotients:

$$\frac{f_1(t)}{t} = 1, \quad \frac{f_2(t)}{t} = \frac{1}{(1-t)^2}, \quad \frac{g_1(t)}{t} = 1, \quad \frac{g_2(t)}{t} = \frac{1}{1-t}.$$

Now, let $\tau = \frac{\pi}{6}$ (spiral-like parameter). One can verify that these functions belong to the spiral-like class $R_m(c, \delta_i, \tau)$ with appropriate parameters δ_i. For the Koebe function f_2, it is known that:

$$e^{i\tau}\frac{zf_2'(z)}{f_2(z)} = e^{i\tau}\frac{1+z}{1-z},$$

and $\mathfrak{R}\left(e^{i\tau}\frac{1+z}{1-z}\right) > 0$ for $|z| < 1$ when $|\tau| < \pi/2$. With $\tau = \pi/6$, this condition holds, so we can take $\delta_2 = 0$. For the starlike function g_2, we have $\frac{zg_2'(z)}{g_2(z)} = \frac{1}{1-z}$, and

$$\mathfrak{R}\left(e^{i\tau}\frac{1}{1-z}\right) > \frac{1}{2}cos(\tau) = \frac{\sqrt{3}}{4} \approx 0.433,$$

so we can take $\delta_2 = \frac{\sqrt{3}}{4}$. Assuming the minimum of these δ_i values is $\delta = \frac{\sqrt{3}}{4}$, then:

$$\gamma = [1-1] + 1 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}.$$

The integral operator F becomes:

$$F_{\alpha_i,\beta_i}(z) = \int_0^z [1]^{\frac{1}{4}}\left[\frac{1}{(1-t)^2}\right]^{\frac{1}{4}}[1]^{\frac{1}{4}}\left[\frac{1}{1-t}\right]^{\frac{1}{4}}dt = \int_0^z (1-t)^{-\frac{1}{2}}(1-t)^{-\frac{1}{4}}dt = \int_0^z (1-t)^{-\frac{3}{4}}dt.$$

Evaluating the integral:

$$F_{\alpha_i,\beta_i}(z) = \left[\frac{(1-t)^{1-\frac{3}{4}}}{1-\frac{3}{4}}\right]_{t=0}^{t=z} = 4\left[1-(1-z)^{\frac{1}{4}}\right].$$

By Theorem 2.3, $F(z) \in V_m(c, \frac{\sqrt{3}}{4}, \frac{\pi}{6})$, meaning it maps the unit disk onto a domain with spiral-like properties (specifically, a domain whose boundary has bounded $\frac{\pi}{6}$-spiral rotation).

3.4. Example 3b: Spiral-like Case (Using Operator H)

Now consider the same parameters $n = 2$, $\alpha_1 = \alpha_2 = \frac{1}{4}$, $\beta_1 = \beta_2 = \frac{1}{4}$, but this time using the derivative-based operator H:

$$H_{\alpha_i,\beta_i}(z) = \int_0^z \prod_{i=1}^{n}[f_i'(t)]^{\alpha_i}[g_i'(t)]^{\beta_i}dt.$$

Choose the following functions:

$$f_1(z) = z \quad \Rightarrow \quad f_1'(t) = 1,$$

$$f_2(z) = \frac{z}{1-z} \quad \Rightarrow \quad f_2'(t) = \frac{1}{(1-t)^2},$$

$$g_1(z) = z \quad \Rightarrow \quad g_1'(t) = 1,$$

$$g_2(z) = \frac{z}{1+z} \quad \Rightarrow \quad g_2'(t) = \frac{1}{(1+t)^2}.$$

Let $\tau = \frac{\pi}{6}$ as before. One can verify that these functions belong to the appropriate spiral-like classes. The integral operator H becomes:

$$H_{\alpha_i,\beta_i}(z) = \int_0^z [1]^{\frac{1}{4}} \left[\frac{1}{(1-t)^2}\right]^{\frac{1}{4}} [1]^{\frac{1}{4}} \left[\frac{1}{(1+t)^2}\right]^{\frac{1}{4}} dt = \int_0^z (1-t)^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}} dt = \int_0^z (1-t^2)^{-\frac{1}{2}} dt.$$

Evaluating the integral:

$$H_{\alpha_i,\beta_i}(z) = arcsin(z).$$

This is the principal branch of the arcsine function, which maps the unit disk onto the infinite strip $\{w : |\Re(w)| < \pi/2\}$.

Remark 3 These examples illustrate the practical utility of our main theorems. The first example verifies the consistency of our results with trivial cases, while the second and third demonstrate how non-trivial functions with specific geometric properties can be constructed systematically. The appearance of special functions (hypergeometric functions) in Example 2 highlights connections between geometric function theory and classical analysis. Moreover, the flexibility in choosing parameters α_i, β_i, and the component functions f_i, g_i allows researchers to tailor the construction to meet specific requirements in applications. This versatility is one of the key strengths of the generalized operators introduced in this work.

3.5. Applications

The results obtained have several practical applications. For instance, these integral operators provide a systematic method for generating conformal mappings with prescribed boundary rotation bounds, useful in computational fluid dynamics and grid generation. The mapping properties established can be used to derive sharp coefficient bounds for functions in these classes. Furthermore, functions constructed via these integral operators often serve as solutions to specific differential equations arising in physics and engineering, particularly in problems involving potential theory and electrostatics.

4. Conclusion

In this paper, we have investigated mapping properties of certain generalized integral operators for classes of functions with bounded boundary rotation of complex order. The main contributions can be summarized as follows:

1. The considered generalized integral operators $F_{\alpha_i,\beta_i}(z)$ and $H_{\alpha_i,\beta_i}(z)$ unify several previously studied operators in geometric function theory.
2. We established sufficient conditions under which these integral operators preserve membership in the classes $V_m(c,\delta)$ and $V_m(c,\delta,\tau)$ of functions with bounded boundary rotation of complex order.
3. Theorems 1 and 3 show that when the component functions f_i, g_i belong to the classes $V_m(c,\delta_i)$ and $R_m(c,\delta_i)$ of bounded boundary and boundary radius rotation, the integral operator $F_{\alpha_i,\beta_i}(z)$ maps to the class $V_m(c,\gamma)$ where γ is a convex combination of the parameters $\alpha_i, \beta_i, \delta_i, (i = 1,2,\cdots,n)$.

4. Theorems 2 and 4 extends these results to the spiral-like case, demonstrating that under appropriate conditions, the integral operator $H_{\alpha_i,\beta_i}(z)$ preserves membership in $V_m(c,\gamma,\tau)$ when the component functions belong to $V_m(c,\delta_i,\tau)$ and $R_m(c,\delta_i,\tau)$.
5. The results obtained are quite general, and through appropriate choices of parameters, yield many known results as special cases, as indicated in Remarks 1 and 2. We would like to mention that these operators can also be studied in Hornich spaces, see e.g., [18].

The techniques employed in this work, primarily based on the convexity properties of the class $P_m(\delta)$, provide a unified approach to studying the mapping properties of various integral operators. The results contribute to the growing body of knowledge in geometric function theory and may find applications in further studies of analytic functions and their geometric properties.

Conflict of Interest

The authors don't have any conflict of interest

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