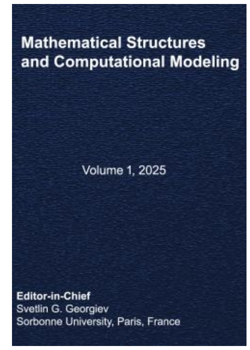
Mathematical Structures and Computational Modeling

ISSN (online): xxxx-xxxx



Asymptotic Inversion of Dirichlet Series with Applications to the Distribution of Prime Numbers

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ARTICLE INFO

Article Type: Research Article

Keywords:
Prime numbers
Number theory
Arithmetic functions

Timeline:
Received: May 18, 2025
Accepted: June 22, 2025
Published: July 12, 2025

Citation: Athanasiou G, Papadopoulos P, Kalkanis K, Psomopoulos C. Asymptotic inversion of dirichlet series with applications to the distribution of prime numbers. Math Struct Comput Model. 2025; 1: 12-37.

DOI: https://doi.org/xx.xxxxx/xxxx-xxxx.2025.1.1

ABSTRACT

This paper investigates the inversion of Dirichlet series, and its applications to the distribution of prime numbers. Several techniques will be developed for inversion using integral kernels, conformal mappings, and asymptotic approximations as arguments approach infinity. These methods are applied to key arithmetic functions, including the prime characteristic function, and prime pair counting function. This paper also analyzes prime gaps, prime k-tuples, and divisor functions, providing asymptotic results and error estimates under the Riemann Hypothesis. The presented techniques are generalizable, offering new insights into prime number behavior and related functions.

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1. Introduction

The study of prime numbers and their distribution is a central problem in analytic number theory. Dirichlet series, introduced by J.P.G.L. Dirichlet, have played a fundamental role in this field due to their deep connections with arithmetic functions and prime number properties. These series provide essential tools for understanding the structure of prime numbers, solving problems in number theory, and analyzing phenomena such as prime gaps, prime pairs, and divisor functions.

The connection between Dirichlet series and prime numbers was first explored through Euler's product formula for the Riemann zeta function. Later, Riemann extended this work by linking the zeros of the zeta function to the distribution of primes, leading to the formulation of the celebrated Riemann Hypothesis. Since then, inversion techniques for Dirichlet series have been pivotal in deriving results related to prime counting functions, prime gaps, and other arithmetic functions.

This paper, systematically investigates methods for inverting Dirichlet series and their asymptotic approximations as the arguments approach infinity. Starting by presenting inversion techniques, including classical methods using integral kernels and advanced methods utilizing conformal mappings. These techniques enable us to approximate the Dirichlet series inversions asymptotically and derive key results in prime number theory.

Using these methods, several applications will be analyzed:

- Prime Characteristic Function: A function that determines whether a number is prime.
- Prime Pair Counting Function: Counting pairs of primes with a fixed gap.
- Prime k-Tuple Function: Generalizing prime pairs to sequences of primes with fixed gaps.
- Prime Gaps and Factor-Counting Functions: Understanding the behavior of gaps between consecutive primes and the number of divisors or prime factors of natural numbers.
- Further asymptotic approximations will be provided for these functions, leveraging the average order theorem and analyzing their behavior under the assumption of the Riemann Hypothesis.
- Additionally, the errors will be evaluated in these approximations and demonstrate their precision.

The results presented here not only enhance our understanding of the distribution of prime numbers but also illustrate the versatility of inversion techniques in solving problems involving arithmetic and analytic functions. These methods can be extended to other classes of series and integral transforms, offering a broader scope for future research.

This paper is organized as follows. In Section 2, techniques for inverting Dirichlet series using integral kernels and conformal mappings will be used. Section 3 focuses on asymptotic approximations of these inversions. In Section 4, various applications are explored to prime number theory, including prime counting functions, prime gaps, and related arithmetic functions. Section 5 provides error estimates for the derived approximations. Finally, a summary of results and possible directions for future research is concluded.

2. Techniques for Inverting the Dirichlet Series

In this section, systematic methods for inverting Dirichlet series are presented, which are central to analytic number theory. The inversion of these series allows us to recover the underlying arithmetic function $f(n)$ from its associated Dirichlet series $D(f,s)$, defined as:

$$D(f,s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad \text{where} s \in \mathbb{C}. \tag{2.0.1}$$

The inversion problem plays a crucial role in deriving properties of arithmetic functions, particularly those connected to the distribution of prime numbers. It begins by introducing integral kernels, which provide a direct approach to inversion, followed by more advanced techniques involving conformal mappings and asymptotic approximations.

2.1. Classical Inversion of the Dirichlet Series

The classical inversion of the Dirichlet series is a cornerstone result in analytic number theory. Given a Dirichlet series of the form

$$D(f,s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s},\quad \Re(s)>\sigma_0, \tag{2.1.1}$$

where $f(n)$ is an arithmetic function and σ_0 is the abscissa of convergence [9], the goal of the inversion is to recover $f(k)$ for any natural number k.

The classical inversion formula, derived using the Mellin transform, is given by the limit:

$$f(k)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}k^{\sigma+it}D(f,\sigma+it)dt \tag{2.1.2}$$

This is actually, a special case of Mellin inversion, or more generally, a special case of Fourier inversion. Using a simple change of variables in equation 2.1.2 takes the following form:

$$f(k)=\lim_{T\to\infty}\frac{1}{2iT}\int_{\sigma-iT}^{\sigma+iT}k^{s}D(f,s)ds. \tag{2.1.3}$$

Where $k\in N$ and $\sigma>\sigma_0$ with σ_0 the abscissa of convergence of the Dirichlet series $D(f,s)$. For a proof of these results, the reader can go to the book [3].

An alternative approach to invert the Dirichlet series involves expressing the solution as a bounded summation, which can be represented in integral form as follows:

$$F(k)=\sum_{1\le n\le k}f(n)=\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{k^s}{s}D(f,s)\frac{ds}{2\pi i}. \tag{2.1.4}$$

According to the books [3] and [5] the arithmetic function $f(k)$ is obtained by the difference:

$$f(k)=F(k)-F(k-1), \tag{2.1.5}$$

With the difference above and equation 2.1.4 the integral below is obtained:

$$f(k)=\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{k^s-(k-1)^s}{s}D(f,s)\frac{ds}{2\pi i}. \tag{2.1.6}$$

2.2. Inversion Using Integral Kernels

Integral kernels provide a powerful method for inverting Dirichlet series, offering an alternative to the classical inversion formula. This approach leverages the idea of constructing a kernel function $K(k,s)$ that allows the Dirichlet series to be expressed in an integral form, facilitating the recovery of the arithmetic function $f(k)$ from its Dirichlet series $D(f,s)$.

The general form of the inversion formula using integral kernels is given by:

$$f(k)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}K(k,s)D(f,s)ds, \tag{2.2.1}$$

where $K(k,s)$ is the integral kernel and $D(f,s)$ is the Dirichlet series of $f(n)$. This method not only provides a direct method of inversion but also gives insight into the asymptotic behavior of the arithmetic function $f(k)$ as $k\to\infty$.

In the following, several specific forms of $K(k,s)$ and their applications will be explored in analytic number theory will be derived. An example of an integral kernel is given by the following expression:

$$K(k,s) = \frac{k^s-(k-1)^s}{s}, \tag{2.2.2}$$

which it is used for the asymptotic inversion of the Dirichlet series. The equation 2.2.2 can alternatively be represented by the contour integral:

$$K(k,s) = \frac{1}{s}\oint_C \frac{z^s}{(z-k)(z-k+1)}\frac{dz}{2\pi i}. \tag{2.2.3}$$

Here, C is a closed contour that encircles the poles at $z = k$ and $z = k - 1$, ensuring that the logarithm of z^s can be properly defined, which leads to a branch cut integral. From 2.2.3, the following operator $\widehat{H}$ is derived:

$$\widehat{H}F(k) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\frac{d^nF(z)}{dz^n}|_{z=k}. \tag{2.2.4}$$

This expression combines elements of the Cauchy integral formula and the geometric series. The constraint here is that the function $F(z)$ must be smooth, meaning $F(z) \in C^{\infty}$. To handle discontinuities, if $F(k)$ can be expressed as $F(\lfloor x \rfloor)$ where $x \in R$, the floor function can be eliminated, allowing the function to be extended to the real numbers via the inclusion map $F(i(\lfloor x \rfloor)) = F(x)$, after which the operator can be applied.

For the specific case $F(z) = z^s$, the kernel $K(k,s)$ takes the form:

$$K(k,s) = \frac{1}{s}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\frac{\Gamma(s+1)k^{s-n}}{\Gamma(s+1-n)}. \tag{2.2.5}$$

The above equation can be derived from equation 2.2.2 by applying the binomial theorem.

Next, the properties of the Dirac delta function is used to construct additional integral kernels. The Dirac delta function is defined by the following integral:

$$\delta(x-a) = \int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^{s-1}}{a^s}\frac{ds}{2\pi i}. \tag{2.2.6}$$

Using this definition, the following equation can be expressed:

$$f(x)\sum_{n=1}^{\infty}\delta(x-n) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}x^{s-1}D(f,s)ds. \tag{2.2.7}$$

To recover $f(k)$, the series is converted as follows:

$$\sum_{n=1}^{\infty}f(n)\delta(x-n) \to \sum_{n=1}^{\infty}f(n)\mathrm{sinc}(k-n) = f(k). \tag{2.2.8}$$

Here, $\mathrm{sinc}(x) = \frac{sin(\pi x)}{\pi x}$ denotes the normalized sine function.

To proceed, the following convolution is being applied:

$$f(k) = \int_0^{\infty}\mathrm{sinc}(k-x)\sum_{n=1}^{\infty}f(n)\delta(x-n)dx, \tag{2.2.9}$$

or equivalently:

$$f(k) = \int_0^{\infty}\mathrm{sinc}(k-x)\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}x^{s-1}D(f,s)dsdx. \tag{2.2.10}$$

By changing the order of integration, the following integral kernel is obtained:

$$K(k,s) = \int_0^{\infty}\mathrm{sinc}(k-x)x^{s-1}dx. \tag{2.2.11}$$

Using a similar approach with some modifications, we derive a comparable integral kernel:

$$K(k,s) = \int_{-\infty}^{\infty} \operatorname{sinc}(x)(k-x)^{s-1}dx. \tag{2.2.12}$$

This kernel will be used in subsequent approximations.

Another approach for constructing integral kernels is to convert Dirichlet series into a different form of series. Using the same techniques as before, the following integral kernels are obtained:

$$\sum_{n=1}^{\infty} f(n)e^{-in\theta} = \int_0^{\infty} e^{-ix\theta} \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{s-1}D(f,s)dsdx. \tag{2.2.13}$$

Changing the order of integration and evaluating this integral using standard techniques [10], we obtain:

$$\int_0^{\infty} e^{-ix\theta}x^{s-1}dx = \frac{\Gamma(s)}{(i\theta)^s}. \tag{2.2.14}$$

Thus, we have the following expression:

$$\sum_{n=1}^{\infty} f(n)e^{-in\theta} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(s)}{(i\theta)^s}D(f,s)ds. \tag{2.2.15}$$

By separating the Fourier series into sine and cosine components, we obtain the following expressions:

$$\sum_{n=1}^{\infty} f(n)\,cos(n\theta) = \int_0^{\infty} cos(\theta x)\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{s-1}D(f,s)dsdx, \tag{2.2.16}$$

$$\sum_{n=1}^{\infty} f(n)\,sin(n\theta) = \int_0^{\infty} sin(\theta x)\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{s-1}D(f,s)dsdx. \tag{2.2.17}$$

By changing the order of integration and using standard results from the theory of integral transforms [10], we find the following identities:

$$\int_0^{\infty} cos(\theta x)x^{s-1}\,dx = \frac{\Gamma(s)}{\theta^s}cos\left(\frac{\pi s}{2}\right), \tag{2.2.18}$$

$$\sum_{n=1}^{\infty} f(n)\,cos(n\theta) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(s)}{\theta^s}cos\left(\frac{\pi s}{2}\right)D(f,s)ds. \tag{2.2.19}$$

$$\int_0^{\infty} sin(\theta x)x^{s-1}dx = \frac{\Gamma(s)}{\theta^s}sin\left(\frac{\pi s}{2}\right), \tag{2.2.20}$$

$$\sum_{n=1}^{\infty} f(n)\,sin(n\theta) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(s)}{\theta^s}sin\left(\frac{\pi s}{2}\right)D(f,s)ds. \tag{2.2.21}$$

Finally, by applying Fourier inversion to the series above, we obtain the following integral kernels:

$$K(k,s) = \int_{-\pi}^{\pi} e^{ik\theta}\frac{\Gamma(s)}{(i\theta)^s}\frac{d\theta}{2\pi}, \quad \Re(s) > 0, \tag{2.2.22}$$

$$K(k,s) = \int_{-\pi}^{\pi} cos(k\theta)\frac{\Gamma(s)}{\theta^s}cos\left(\frac{\pi s}{2}\right)\frac{d\theta}{2\pi}, \quad \Re(s) > 0, \tag{2.2.23}$$

and

$$K(k,s) = \int_{-\pi}^{\pi} sin(k\theta)\frac{\Gamma(s)}{\theta^s}sin\left(\frac{\pi s}{2}\right)\frac{d\theta}{2\pi}, \quad \Re(s) > 0. \tag{2.2.24}$$

In a similar manner, we can obtain the Laurent series representation:

$$\sum_{n=1}^{\infty} f(n)z^{-n} = \int_0^{\infty} z^{-x}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{s-1}D\,(f,s)dsdx. \tag{2.2.25}$$

By changing the order of integration, we deduce:

$$\int_0^\infty e^{-x\, log(z)} x^{s-1} dx = \frac{\Gamma(s)}{(log(z))^s}. \tag{2.2.26}$$

Thus, we arrive at the final expression:

$$\sum_{n=1}^{\infty} f(n) z^{-n} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\Gamma(s)}{(log(z))^s} D(f,s) ds. \tag{2.2.27}$$

By applying Cauchy's integral formula, we obtain the kernel:

$$K(k,s) = \frac{1}{2\pi i} \oint_C z^{k-1} \frac{\Gamma(s)}{(log(z))^s} dz. \tag{2.2.28}$$

where C is a closed contour, such that the integrated function is well-defined on the complex plane.

For all the kernels discussed here, the variable k must be a natural number, i.e., $k \in N$.

2.3. Inverting the Series Using Conformal Mapping

An alternative technique for inverting the Dirichlet series involves the use of conformal mapping. Consider the integral representation for the inversion of a Dirichlet series:

$$f(k) = \lim_{T \to \infty} \frac{1}{4\pi i T} \int_{\sigma - 2\pi i T}^{\sigma + 2\pi i T} k^s D(f,s) ds, \tag{2.3.1}$$

where $\sigma > \sigma_0$, and σ_0 is the abscissa of convergence of the Dirichlet series $D(f,s)$. In this formulation, we introduce a change of variables by replacing T with $2\pi T$ for convenience.

To simplify the evaluation, we divide the integral into segments of height $2\pi i$, effectively splitting the contour into horizontal strips. This yields:

$$\lim_{T \to \infty} \frac{1}{4\pi i T} \int_{\sigma - 2\pi i T}^{\sigma + 2\pi i T} k^s D(f,s) ds = \lim_{T \to \infty} \sum_{n=-T}^{T-1} \frac{1}{4\pi i T} \int_{\sigma + 2\pi i n}^{\sigma + 2\pi i (n+1)} k^s D(f,s) ds. \tag{2.3.2}$$

At this stage, we employ a conformal mapping via the exponential function $exp(s) = e^s$, which maps the complex plane C as follows:

$$exp: C \to (0, +\infty) \times R/2\pi Z.$$

The inverse of this mapping is the logarithmic function $log(s)$, where the real axis of C maps onto a spiral-like surface with periodicity 2π in the imaginary direction. Explicitly, $R/2\pi Z$ denotes the angle periodicity in the exponential mapping, where $2\pi Z = \{2\pi m : m \in Z\}$.

Using this transformation, the integral can be expressed on the exponential contour $|z| = e^\sigma$ as follows:

$$\lim_{T \to \infty} \sum_{n=-T}^{T-1} \frac{1}{4\pi i T} \oint_{|z|=e^\sigma} \frac{k^{\log(z)}}{z} D(f, \log(z)) dz. \tag{2.3.3}$$

Simplifying further by noting that the sum over n contributes a factor of $2T$, we obtain:

$$\lim_{T \to \infty} \frac{2T}{4\pi i T} \oint_{|z|=e^\sigma} \frac{k^{\log(z)}}{z} D(f, \log(z)) dz. \tag{2.3.4}$$

In simpler terms, as $T \to \infty$, the above expression reduces to:

$$f(k) = \frac{1}{2\pi i} \oint_{|z|=e^\sigma} \frac{k^{log(z)}}{z} D(f, log(z)) dz. \tag{2.3.5}$$

The above result provides an alternative inversion formula for the Dirichlet series, with the contour integral taken over the curve $|z| = e^{\sigma}$. However, this result can be extended to any closed curve C in the complex plane, provided that the logarithm of the integrated function $k^{log(z)}$ is well-defined (i.e., avoiding branch cuts). The generalized inversion formula is therefore:

$$f(k) = \frac{1}{2\pi i} \oint_C \frac{k^{log(z)}}{z} D(f, log(z)) dz. \tag{2.3.6}$$

Here, C is a contour that encloses the relevant singularities of the transformed series $D(f, log(z))$, ensuring that the integral converges and the logarithmic function remains single-valued within the region.

2.4. Asymptotic Approximations

In this subsection, we derive asymptotic approximations for the integral kernels $K(k, s)$ as $k \to \infty$. These approximations are central to simplifying the inversion of Dirichlet series and are particularly useful in applications requiring asymptotic analysis.

We begin with the following integral kernel:

$$K(k, s) = \int_{-\pi}^{\pi} e^{ik\theta} \frac{\Gamma(s)}{(i\theta)^s} \frac{d\theta}{2\pi}. \tag{2.4.1}$$

By applying the Riemann-Lebesgue lemma [12], as $k \to +\infty$, the oscillatory behavior of the exponential term dominates, leading to the approximation:

$$K(k, s) \sim \frac{\Gamma(s)}{\pi^{s+1} k} sin\left(k\pi - s\frac{\pi}{2}\right). \tag{2.4.2}$$

Next, we consider an alternative integral kernel given by:

$$K(k, s) = \frac{1}{2\pi i} \oint_C z^{k-1} \frac{\Gamma(s)}{(log(z))^s} dz, \tag{2.4.3}$$

where C is a contour enclosing the singularity at $z = 1$. Using the final value theorem for the Z-transform [10], we obtain the leading-order behavior:

$$K(k, s) \sim \lim_{z \to 1} (z - 1) \frac{\Gamma(s)}{(log(z))^s} \tag{2.4.4}$$

By applying the definition of the derivative near $z = 1$, we simplify the above expression to:

$$K(k, s) \sim \frac{\Gamma(s)}{s(log(z))^{s-1}}, \quad z \to 1. \tag{2.4.5}$$

The above result provides the asymptotic form for the kernel near its dominant singularity. Next, we examine two additional forms of integral kernels, which asymptotically converge to the same result. The first form is given by:

$$K(k, s) = \frac{1}{s} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{\Gamma(s+1) k^{s-n}}{\Gamma(s+1-n)}, \tag{2.4.6}$$

while the second form is expressed as:

$$K(k, s) = \int_{-\infty}^{\infty} \mathrm{sinc}(x) (k - x)^{s-1} dx, \tag{2.4.7}$$

where $\mathrm{sinc}(x) = \frac{sin(\pi x)}{\pi x}$ is the normalized sine function.

Both kernels asymptotically simplify to the same leading-order term as $k \to \infty$:

$$K(k,s) \sim k^{s-1}. \tag{2.4.8}$$

To justify this result, consider the first integral kernel. Since the exponents of k in the series form a decreasing sequence, the leading-order behavior is dominated by the first term of the summation. Thus, the approximation reduces to k^{s-1}.

For the second kernel, we proceed as follows. Factoring out k^{s-1} from the integral, we write:

$$K(k,s) = k^{s-1} \int_{-\infty}^{\infty} \mathrm{sinc}(x) \left(1 - \frac{x}{k}\right)^{s-1} dx. \tag{2.4.9}$$

As $k \to \infty$, the term $\left(1 - \frac{x}{k}\right)^{s-1}$ can be approximated by:

$$\left(1 - \frac{x}{k}\right)^{s-1} \sim 1.$$

Substituting this approximation back into the integral, we obtain:

$$K(k,s) \sim k^{s-1} \int_{-\infty}^{\infty} \mathrm{sinc}(x) dx. \tag{2.4.10}$$

Using the standard result:

$$\int_{-\infty}^{\infty} \mathrm{sinc}(x) dx = 1,$$

we recover the asymptotic form:

$$K(k,s) \sim k^{s-1}. \tag{2.4.11}$$

This result demonstrates that both integral kernels converge asymptotically to the same leading-order behavior. Due to its simplicity and generality, this approximation will be utilized extensively in subsequent applications involving the asymptotic inversion of Dirichlet series.

2.5. Applications

In this subsection, we apply the previously derived asymptotic approximations to invert the Dirichlet series and approximate the corresponding arithmetic functions. The key tool we will use is the following Average Order Theorem, which has wide-ranging applications.

Theorem 1 (Average Order Theorem) Let $S_x = \sum_{n \le x} a_n$ represent the sum of an arithmetic function a_n, and suppose that S_x can be asymptotically approximated by a differentiable function $f(x) \in C^1$, where $f(x) \to \infty$ as $x \to \infty$. Then, under the condition:

$$S_x = f(x) + O(f'(x)), \quad x \to \infty,$$

the individual terms a_k can be approximated as:

$$a_k \sim \left.\frac{dS_x}{dx}\right|_{x=k} \sim \left.\frac{df(x)}{dx}\right|_{x=k}, \quad k \to \infty.$$

Proof. We start with the sum:

$$S_x = \sum_{n \le x} a_n. \tag{2.5.1}$$

By assumption, S_x can be asymptotically approximated by a smooth function $f(x)$, such that:

$$S_x \sim f(x), \quad x \to \infty. \tag{2.5.2}$$

The sequence a_n is then approximated by the derivative of $f(x)$:

$$a_k \sim \left.\frac{df(x)}{dx}\right|_{x=k}, \quad k \to \infty. \tag{2.5.3}$$

To justify this result, we invoke the inversion formula for the Dirichlet series:

$$S_x = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{s} D(a,s) \frac{ds}{2\pi i} : f(x), \tag{2.5.4}$$

where:

$$D(a,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \tag{2.5.5}$$

Substituting the kernel approximation $K(k,s): k^{s-1}$ leads to:

$$a_k \sim \int_{\sigma-i\infty}^{\sigma+i\infty} k^{s-1} D\,(a,s) \frac{ds}{2\pi i} = \left.\frac{dS_x}{dx}\right|_{x=k}. \tag{2.5.6}$$

Finally, applying De l'Hospital's rule confirms the asymptotic approximation in (2.5.3).

Handling Discontinuities with the Dirac Delta Function: If $f(x)$ and S_x exhibit Dirac delta-type discontinuities at integer points, we substitute the Dirac delta function with the normalized sine function $\mathrm{sinc}(k-a)$, defined as:

$$\delta(k-a) \to \frac{sin(\pi(k-a))}{\pi(k-a)} = \mathrm{sinc}(k-a), \quad k, a \in \mathrm{Z}. \tag{2.5.7}$$

This substitution is valid only at integer discontinuities and is justified using the kernel approximation derived earlier.

Nowhere Differentiable Functions: If $f(x)$ and S_x are nowhere differentiable, the arithmetic function a_k can be approximated using its mean value:

$$a_k \sim \frac{S_k}{k} \sim \frac{f(k)}{k}, \quad k \to \infty. \tag{2.5.8}$$

This result follows directly from the Average Order Theorem by applying De l'Hospital's rule.

Connection of the Average Order Theorem with the Mean Value Theorem: As observed, the average order theorem resembles a kind of mean value theorem. This can be explained by considering the differential mean value theorem on the interval $(N-1, N)$ as $N \to \infty$, specifically:

$$f'(\xi) = \frac{f(N) - f(N-1)}{N-(N-1)}, \quad \xi \in (N-1, N).$$

Since $N \to \infty$, we can approximate $\xi \approx N$, which gives:

$$f'(N) \sim f(N) - f(N-1).$$

Furthermore, the average order theorem resolves the issue of discontinuities of the form $f(N) - f(N-1) = \infty - \infty$ for $f(N) \to \infty$ as $N \to \infty$.

2.6. Specific Applications of the Average Order Theorem

2.6.1. Zeta Function and Divisor Sums

Consider the zeta function:

$$\zeta(s) = D(1, s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx, \tag{2.6.1}$$

where $\{x\}$ represents the fractional part of x. Using the convolution property of Dirichlet series:

$$D(f, s)D(g, s) = D(f * g, s), \tag{2.6.2}$$

where:

$$f * g = \sum_{k|n} f(k)g\left(\frac{n}{k}\right), \tag{2.6.3}$$

and substituting $g\left(\frac{n}{k}\right) = 1$, we obtain:

$$\zeta(s)D(f, s) = \sum_{n=1}^{\infty} \frac{\sum_{k|n} f(k)}{n^s}. \tag{2.6.4}$$

Applying the Average Order Theorem leads to the approximation:

$$\sum_{k|n} f\,(k) \sim \int_{\sigma-i\infty}^{\sigma+i\infty} n^{s-1}\zeta\,(s)D(f, s)\frac{ds}{2\pi i} \sim \sum_{m=1}^{n} \frac{f(m)}{m}, \quad n \to \infty. \tag{2.6.5}$$

2.6.2. Divisor Characteristic Function

The characteristic function for divisors $1_{m|n}$ has the Dirichlet series:

$$\frac{1}{m^s}\zeta(s) = \sum_{n=1}^{\infty} \frac{1_{m|n}}{n^s}. \tag{2.6.6}$$

The divisor characteristic function $1_{m|n}$ is defined as:

$$1_{m|n} = \begin{cases} 1, & \text{if} m|n, \\ 0, & \text{otherwise.} \end{cases}$$

Using similar steps, we obtain the approximation:

$$1_{m|n} \sim \frac{u(n-m)}{m}, \quad n \to \infty, \tag{2.6.7}$$

where $u(x)$ is the Heaviside step function.

2.6.3. Greatest Common Divisor Indicator Function

The greatest common divisor (gcd) indicator function is defined as:

$$1_{gcd(k,n)=m} = \begin{cases} 1, & \text{if } gcd(\,k, n) = m, \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding Dirichlet series is:

$$\zeta(s)\frac{1_{m|k}}{m^s}\prod_{p|\frac{k}{m}}\left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{1_{gcd(k,n)=m}}{n^s}.$$

Using the average order theorem, we derive:

$$1_{gcd(k,n)=m} \sim u(n-m)\frac{1_{m|k}}{m}\prod_{p|\frac{k}{m}}\left(1 - \frac{1}{p}\right),$$

or equivalently:

$$1_{gcd(k,n)=m} \sim u(n-m)\frac{1_{m|k}}{k}\varphi\left(\frac{k}{m}\right),$$

where $\varphi(n)$ is the Euler totient function, defined by:

$$\varphi(n) = n\prod_{p|n}\left(1-\frac{1}{p}\right) \tag{2.6.8}$$

2.6.4. Prime Characteristic Function and the Prime Number Theorem

The prime characteristic function $\chi_P(n)$ is defined as:

$$\chi_P(n) = \begin{pmatrix} 1, & n \in P, \\ 0, & n \notin P. \end{pmatrix} \tag{2.6.9}$$

The prime counting function $\pi(x)$ is given by:

$$\pi(x) = \sum_{1\le n\le x} \chi_P\,(n). \tag{2.6.10}$$

According to the Prime Number Theorem [?], we have:

$$\pi(x) \sim \int_2^x \frac{1}{log\, t}dt, \quad x \to \infty. \tag{2.6.11}$$

By the Average Order Theorem, the prime characteristic function satisfies:

$$\chi_P(n) \sim \frac{u(n-2)}{log\, n}, \quad n \to \infty. \tag{2.6.12}$$

2.6.5. Prime Number Function

Another application, is finding a function that generates prime numbers, or the inverse prime counting function. That is defined as:

$$\pi^{-1}: N \to P$$

or, in another notation:

$$\pi^{-1}(n) = p_n$$

where p_n is the nth prime number.

With the corresponding Dirichlet series:

$$D(\pi^{-1}, s) = \sum_{n=1}^{\infty} \frac{\pi^{-1}(n)}{n^s}. \tag{2.6.13}$$

An alternative presentation of this series is the following:

$$D(\pi^{-1}, s) = \sum_{p\in P} \frac{p}{\pi(p)^s}, \tag{2.6.14}$$

with P the set of the prime numbers and $\pi(n)$ is the prime counting function [4]. Another presentation of the series above, is by Stieltjes integration [7] with measure the prime counting function, yields:

$$D(\pi^{-1}, s) = \int_2^{\infty} \frac{t}{\pi(t)^s}\frac{d\pi(t)}{dt}dt. \tag{2.6.15}$$

Using integration by parts, gives the following result:

$$D(\pi^{-1}, s) = \frac{2}{s-1} + \frac{1}{s-1}\int_{2+\varepsilon}^{\infty} \frac{1}{\pi(t)^{s-1}}dt, \ \forall\varepsilon \in (0,1). \tag{2.6.16}$$

Approximating the inversion of the Dirichlet series, we get:

$$\pi^{-1}(k) \sim \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} k^{s-1} D\,(\pi^{-1}, s)ds. \tag{2.6.17}$$

or:

$$\pi^{-1}(k) = 2u(k-1) + \int_{2+\varepsilon}^{\infty} u\,(k-\pi(t))dt,\ \forall\varepsilon \in (0,1). \tag{2.6.18}$$

Or expressing the above equation in a series form:

$$\pi^{-1}(k) = 2u(k-1) + \sum_{n=2}^{\infty} u\,(k-\pi(n)-1). \tag{2.6.19}$$

The inverse prime counting function $\pi^{-1}(k)$, can be expressed in terms of the prime gap function $g_k = \pi^{-1}(k+1) - \pi^{-1}(k)$ 2.6.16 gets the form:

$$D(\pi^{-1}, s) = \frac{2}{s-1} + \frac{1}{s-1}\sum_{n=1}^{\infty}\int_{p_n+\varepsilon}^{p_{n+1}+\varepsilon} \frac{1}{(n+1)^{s-1}} dt\,,\ \forall\varepsilon \in (0,1), \tag{2.6.20}$$

or:

$$D(\pi^{-1}, s) = \frac{2}{s-1} + \frac{1}{s-1}\sum_{n=1}^{\infty}\frac{g_n}{(n+1)^{s-1}}. \tag{2.6.21}$$

Using the approximate Dirichlet series inversion:

$$\pi^{-1}(k) = 2u(k-1) + \sum_{n=1}^{\infty} g_n u\,(k-n-1). \tag{2.6.22}$$

2.6.6. Prime Gap Function

According to the previous results, prime gap function is given by the equation:

$$g_k = \sum_{n=0}^{\infty} sinc\,(k-\pi(n)){:}\frac{d\pi^{-1}(x+1)}{dx}|_{x=k},\ k \in N. \tag{2.6.23}$$

This function g_k, is defined as the difference between consecutive primes p_{k+1} and p_k.

2.6.7. Prime Factor Counting Function

The prime factor counting function $\omega(n)$, which counts the number of distinct prime factors of n, satisfies the relation:

$$\omega(n) = \sum_{k|n}\chi_P\,(k). \tag{2.6.24}$$

With a combination of the results of this subsection, for $n \to \infty$ yields to the following asyptotic approximation:

$$\omega(n) \sim \sum_{m=1}^{n}\frac{\chi_P(m)}{m} \sim \sum_{m=2}^{n}\frac{1}{mlog(m)},\ n \to \infty. \tag{2.6.25}$$

or, using the Euler-Maclaurin summation approximation [?]:

$$\omega(n) \sim \sum_{m=2}^{n}\frac{1}{mlog(m)} \sim \int_{2}^{n}\frac{1}{tlog(t)}\ dt = log(log(n)) - log(log(2)),\ n \to \infty. \tag{2.6.26}$$

which is a proof of the Hardy-Ramanujan theorem [5].

2.6.8. Divisor Counting Function

The divisor counting function $d(n)$, which counts the number of divisors of n, satisfies:

$$d(n) = \sum_{k|n} 1. \tag{2.6.27}$$

Which can be approximated by the harmonic series H_n:

$$d(n) \sim \sum_{m=1}^{n} \frac{1}{m} = H_n, \; n \to \infty. \tag{2.6.28}$$

Using the average order theorem approach, consider the function:

$$\sigma(x) = \sum_{n \le x} d(n). \tag{2.6.29}$$

The function $\sigma(x)$ can be approximated as in the book ([5] by the expression:

$$\sigma(x) \sim x log(x) + (2\gamma - 1)x. \tag{2.6.30}$$

As observed, the function diverges to infinity as x approaches infinity, using the average order theorem to derive $d(n)$:

$$d(n) \sim log(n) + 2\gamma, \; n \to \infty, \tag{2.6.31}$$

where γ is the Euler-Mascheroni constant.

3. Prime Numbers

In this section, we introduce key functions that describe and quantify the distribution of prime numbers. These functions are fundamental tools in analytic number theory and provide insights into the properties of primes. Additionally, we examine their asymptotic approximations as their arguments approach infinity. A significant aspect of this discussion involves the role of the Riemann zeta function and its connection to the distribution of prime numbers. In particular, the location of the roots of the zeta function and the Riemann Hypothesis play a central role in determining the accuracy of these approximations.

The Riemann Hypothesis conjectures that all nontrivial zeros of the zeta function lie on the critical line $\Re(s) = \frac{1}{2}$ in the complex plane. If true, it provides deep implications for the error terms in approximations of prime-related functions and, consequently, for the distribution of primes.

In the following subsections, we explore key arithmetic functions related to prime numbers, such as the prime characteristic function, the prime pair counting function, and the prime gap function. We derive their asymptotic forms using inversion techniques for Dirichlet series, integral kernels, and results from asymptotic analysis.

3.1. Prime Characteristic Function

The prime characteristic function is a fundamental arithmetic function that identifies whether a given natural number n is prime. It is defined as:

$$\chi_P(n) = \begin{cases} 1, & \text{if} n \in \mathrm{P}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.1.1}$$

where P denotes the set of prime numbers.

The prime characteristic function can be expressed as the following integral:

$$\chi_{\mathrm{P}}(n) = \lim_{T \to \infty} \frac{1}{2iT} \int_{\sigma - iT}^{\sigma + iT} n^s P(s) ds, \tag{3.1.2}$$

where $\sigma > 1$ and $P(s)$ is a Dirichlet series given by:

$$P(s) = \sum_{p \in P} \frac{1}{p^s}. \tag{3.1.3}$$

The Dirichlet series $P(s)$ is closely related to the Riemann zeta function $\zeta(s)$, as shown in [4]. It can be expressed in terms of the Mobius function $\mu(m)$ as:

$$P(s) = \sum_{m=1}^{\infty} \frac{\mu(m)\, log(\zeta(ms))}{m}. \tag{3.1.4}$$

The Mobius Function: The Mobius function $\mu(n)$ is an important multiplicative function in number theory and is defined as follows:

$$\mu(n) = \begin{cases} (-1)^{\omega(n)}, & \text{if } p^2 \nmid n \forall p \in P, \\ 1, & \text{if } n = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{3.1.5}$$

where $\omega(n)$ is the number of distinct prime factors of n.

The Mobius function satisfies the following key properties:

- **The Mobius inversion formula:**

$$\delta_{n,1} = \sum_{d|n} \mu\,(d), \tag{3.1.6}$$

where $\delta_{n,1}$ is the Kronecker delta function.

- **The Dirichlet series representation of $\boldsymbol{\mu(n)}$:**

$$\frac{1}{\zeta(s)} = D(\mu, s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \Re(s) > 1. \tag{3.1.7}$$

Deriving a Formula for the Prime Characteristic Function: Substituting equation (3.1.4) into the integral representation (3.1.2) and performing a change of variables, we obtain the following result for $\chi_P(n)$:

$$\chi_{\mathrm{P}}(n) = \lim_{T\to\infty} \frac{1}{2iT} \int_{\sigma-iT}^{\sigma+iT} \sum_{m=1}^{\infty} \frac{\mu(m)}{m} n^{\frac{s}{m}} \log(\zeta(s))ds. \tag{3.1.8}$$

The logarithm of the Riemann zeta function $\zeta(s)$ can be expanded as:

$$log(\,\zeta(s)) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{log(n)} \frac{1}{n^s}, \tag{3.1.9}$$

where $\Lambda(n)$ is the von Mangoldt function, which is defined as:

$$\Lambda(n) = \begin{cases} log(\,p), & \text{if } n = p^r \text{ for some prime } p \text{ and integer } r \geq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.10}$$

For further details regarding the Mobius function $\mu(m)$ and the von Mangoldt function $\Lambda(n)$, readers are referred to [5].

We can compute the prime characteristic function $\chi_P(n)$ using a summation involving the von Mangoldt function $\Lambda(n)$ and the Mobius function $\mu(m)$. The function is expressed as follows:

$$\chi_P(n) = \sum_{n=k^m} \frac{\Lambda(k)}{log(k)} \frac{\mu(m)}{m}, \tag{3.1.11}$$

where the summation is taken over all integers k and $m \geq 1$ such that $n = k^m$. This expression can be rewritten in a more convenient form:

$$\chi_P(n) = \frac{1}{log(n)} \sum_{n=k^m} \Lambda(k)\mu\,(m). \tag{3.1.12}$$

In a more compact notation, equation (3.1.12) can be expressed as:

$$\chi_{\mathrm{P}}(n)=\frac{1}{\log(n)}\sum_{\substack{m\geq 1\\ n^{1/m}\in\mathrm{N}}}\Lambda(n^{1/m}\mu(m), \tag{3.1.13}$$

where the summation is taken over all $m \geq 1$ such that $n^{1/m}$ is a natural number. This compact form emphasizes the role of the Mobius function $\mu(m)$ in eliminating contributions from powers of primes.

Von Mangoldt Function and the Zeta Function Roots: According to Landau [1], the von Mangoldt function $\Lambda(n)$ can be expressed as a summation involving the nontrivial zeros ρ of the Riemann zeta function $\zeta(s)$ within the critical strip $0 < \Re(\rho) < 1$. Specifically, we have:

$$\Lambda(n)=\lim_{T\to\infty}-\frac{\pi}{T}\sum_{\substack{\zeta(\rho)=0\\ -T\leq\Im(\rho)\leq T\\ 0<\Re(\rho)<1}} n^{\rho}. \tag{3.1.14}$$

Here, the sum is taken over all nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$, where β and γ denote the real and imaginary parts of ρ, respectively. This expression highlights the deep connection between the von Mangoldt function and the distribution of the zeros of the zeta function, which, in turn, underpins the distribution of prime numbers.

By changing the order of summation and applying the Taylor expansion for the exponential function, we obtain the following result:

$$F(n;s) = \sum_{m=1}^{\infty} \mu(m) n^{\frac{s}{m}} = \sum_{k=1}^{\infty} \frac{log(n)^k}{k!\zeta(k)} s^k. \tag{3.1.15}$$

Using the integral representation in (3.1.8), the prime characteristic function $\chi_P(n)$ can be expressed as:

$$\chi_{\mathrm{P}}(n)=\frac{1}{\log(n)}\sum_{k=1}^{\infty}\sum_{m=2}^{\infty}\Lambda(m)\frac{\log(n)^k}{k!\zeta(k)}\lim_{T\to\infty}\frac{1}{2iT}\int_{\sigma-iT}^{\sigma+iT}\frac{s^k}{m^s}ds. \tag{3.1.16}$$

Using the Dirichlet series for the logarithmic derivative of the Riemann zeta function:

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}, \tag{3.1.17}$$

we can rewrite $\chi_P(n)$ as:

$$\chi_{\mathrm{P}}(n)=\frac{1}{\log(n)}\sum_{k=1}^{\infty}\frac{\log(n)^k}{k!\zeta(k)}\lim_{T\to\infty}\frac{1}{2iT}\int_{\sigma-iT}^{\sigma+iT}s^k(-\frac{\zeta'(s)}{\zeta(s)})ds. \tag{3.1.18}$$

Alternatively, this can be written in a more compact form:

$$\chi_{\mathrm{P}}(n)=\frac{1}{\log(n)}\lim_{T\to\infty}\frac{1}{2iT}\int_{\sigma-iT}^{\sigma+iT}F(n;s)(-\frac{\zeta'(s)}{\zeta(s)})ds. \tag{3.1.19}$$

Using the fact that $\inf_{n\in\mathrm{Z}^+}\zeta(n)=1$ and applying the Taylor expansion for the exponential function, we derive the following inequality:

$$0 \leq \chi_P(n) \leq \frac{\Lambda(n)}{log(n)}. \tag{3.1.20}$$

Furthermore, since $\inf_{n\in\mathrm{Z}^+}\zeta(n)=1$, we obtain:

$$|F(n;s)| < |n^s|.$$

Connection to the Zeta Function Zeros: According to Landau [1], the prime characteristic function can also be expressed as a summation involving the nontrivial zeros ρ of the Riemann zeta function $\zeta(s)$. Specifically, we have:

$$\chi_{\mathsf{P}}(n) = \lim_{T\to\infty} -\frac{\pi}{\log(n)T} \sum_{\substack{\zeta(\rho)=0 \\ -T\leq\Im(\rho)\leq T \\ 0<\Re(\rho)<1}} F(n;\rho), \tag{3.1.21}$$

where the summation is taken over all nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$, with $0 < \Re(\rho) = \beta < 1$ and $\Im(\rho) = \gamma$. With upper bound:

$$\limsup_{n\to\infty} \chi_{\mathsf{P}}(n) = \limsup_{n\to\infty} \frac{\Lambda(n)}{log(n)} = 1. \tag{3.1.22}$$

Returning to the definition of the prime counting function, it is expressed as:

$$\pi(x) = \sum_{1\leq n\leq x} \chi_P(n), \tag{3.1.23}$$

where $\chi_P(n)$ is the prime characteristic function. Using the operator $\widehat{H}$ introduced in (2.2.5), the prime characteristic function $\chi_P(x)$ can be written as:

$$\chi_P(x) = \widehat{H}(\pi(x)). \tag{3.1.24}$$

The operator $\widehat{H}$ is defined as:

$$\widehat{H}(\pi(x)) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{d^n \pi(x)}{dx^n}. \tag{3.1.25}$$

Another Explicit Formula for the Prime Characteristic Function: As described in [4], the prime counting function $\pi(x)$ can be expressed explicitly as:

$$\pi(x) = \sum_{m\geq 1} \frac{\mu(m)}{m} \mathrm{li}(x^{1/m}) - \sum_{\zeta(\rho)=0} \sum_{m\geq 1} \frac{\mu(m)}{m} \mathrm{li}(x^{\rho/m}), \tag{3.1.26}$$

where $\mathrm{li}(x)$ is the logarithmic integral function, defined as:

$$\mathrm{li}(x) = \int_2^x \frac{1}{log(t)} dt. \tag{3.1.27}$$

To construct $\chi_P(x)$ from the operator $\widehat{H}$ is used by the auxiliary function:

$$R(x^s) = \sum_{m\geq 1} \frac{\mu(m)}{m} \mathrm{li}(x^{s/m}). \tag{3.1.28}$$

Using the operator $\widehat{H}$, we define:

$$Q(x;s) = \widehat{H}(\mathrm{li}(x^s)). \tag{3.1.29}$$

To simplify differentiation, consider the derivative of $Q(x;s)$ with respect to s:

$$\frac{dQ(x;s)}{ds} = \widehat{H}\left(\frac{d\mathrm{li}(x^S)}{ds}\right). \tag{3.1.30}$$

Since:

$$\frac{d\mathrm{li}(x^S)}{ds} = \frac{x^S}{s}, \tag{3.1.31}$$

applying $\widehat{H}$ yields:

$$\frac{dQ(x;s)}{ds} = \widehat{H}\left(\frac{x^s}{s}\right). \tag{3.1.32}$$

Expressing this in series form:

$$\frac{dQ(x;s)}{ds} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{1}{s} \frac{d^n x^s}{dx^n}. \tag{3.1.33}$$

Using the derivatives of the power function x^s, this becomes:

$$\frac{dQ(x;s)}{ds} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{\Gamma(s)}{\Gamma(s-n+1)} x^{s-n}. \tag{3.1.34}$$

To compute $Q(x;s)$, integrate with respect to s:

$$Q(x;s) = \int_{-\infty}^{s} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{\Gamma(t)}{\Gamma(t-n+1)} x^{t-n}\, dt. \tag{3.1.35}$$

Combining the previous results, the prime characteristic function $\chi_P(k)$ can be expressed as:

$$\chi_P(k) = \sum_{m\geq 1} \frac{\mu(m)}{m} Q\,(k; 1/m) - \sum_{\zeta(\rho)=0} \sum_{m\geq 1} \frac{\mu(m)}{m} Q(k; \rho/m), \tag{3.1.36}$$

where $\zeta(\rho) = 0$ denotes the zeros of the Riemann zeta function.

3.1.1. Asymptotic Approximation

Using the average order theorem, the prime characteristic function $\chi_P(n)$ admits the following asymptotic expansion:

$$\chi_P(n) \sim \frac{u(n-2)}{log(n)} + O(1), \tag{3.1.37}$$

where $u(n)$ is the unit step function and O denotes the Landau big-O notation as described in [12].

Under the assumption of the Riemann Hypothesis, this approximation can be refined to:

$$\chi_P(n) = \frac{u(n-2)}{log(n)} + O\left(\frac{log(n)}{\sqrt{n}}\right). \tag{3.1.38}$$

Ramanujan's Approximation: An alternative asymptotic approximation for the prime characteristic function was provided by Ramanujan, as detailed in [2]. This approximation is given by:

$$\chi_P(n) \sim \frac{d\pi(x)}{dx}\bigg|_{x=n} \sim \frac{1}{n\,log(n)} \sum_{m\geq 1} \frac{\mu(m)}{m} n^{1/m}. \tag{3.1.39}$$

Using the series expansion for $F(n;s)$ from (3.1.15), this can be rewritten as:

$$\chi_P(n) \sim \frac{1}{n\,log(n)} \sum_{k=0}^{\infty} \frac{log(n)^k}{k!\zeta(k+1)}. \tag{3.1.40}$$

Notes on Asymptotics: The asymptotic expansions presented above highlight the central role of the logarithmic term $log(n)$ and the contribution of higher-order corrections. The refined result incorporating the Riemann Hypothesis emphasizes the connection between prime-related functions and the zeros of the Riemann zeta function, further illustrating the deep interplay between prime number distribution and analytic number theory.

3.2. Prime Pair Counting Function

The problem considered in this subsection is the construction of a function that counts pairs of prime numbers with a given integer gap g. Specifically, we k to count pairs of prime numbers (p, q) such that $p + g = q$ with $p, q \in P$, where P denotes the set of all prime numbers.

Indicator Function for Prime Pairs: We begin by defining an indicator function $1_E(n)$, which identifies numbers n such that both n and $n-g$ are prime:

$$1_E(n) = 1_P(n)1_P(n-g), \tag{3.2.1}$$

where the set E is defined as:

$$E = \{n \in P\} \cap \{n - g \in P\}.$$

The general form of an indicator function $1_\Omega(x)$ is given as:

$$1_\Omega(x) = \begin{cases} 1, & \text{if} x \in \Omega, \\ 0, & \text{if} x \notin \Omega. \end{cases} \tag{3.2.2}$$

Using properties of indicator functions from measure theory [11], we replace $1_P(n)$ with the prime characteristic function $\chi_P(n)$, yielding:

$$1_E(n) = \chi_P(n)\chi_P(n-g). \tag{3.2.3}$$

Definition of the Prime Pair Counting Function: The prime pair counting function with gap g, denoted $\pi_g(x)$, is then defined as:

$$\pi_g(x) = \sum_{n\le x} \chi_P(n)\chi_P(n-g). \tag{3.2.4}$$

Using the inequality (3.1.20), we derive an upper bound for $\pi_g(x)$:

$$\pi_g(x) \le \sum_{2+g\le n\le x} \frac{\Lambda(n)}{log(n)} \frac{\Lambda(n-g)}{log(n-g)}, \tag{3.2.5}$$

where $\Lambda(n)$ is the von Mangoldt function. This upper bound is particularly useful for testing prime gap conjectures, as it indicates whether the sum converges or diverges.

Asymptotic Approximation: From inequality (3.2.5), the asymptotic approximation for $\pi_g(x)$ is:

$$\pi_g(x) \sim \sum_{2+g\le n\le x} \frac{\Lambda(n)}{log(n)} \frac{\Lambda(n-g)}{log(n-g)}. \tag{3.2.6}$$

Using the asymptotic expansion of the prime characteristic function (3.1.37), we deduce:

$$\pi_g(x) \sim \sum_{2+g\le n\le x} \frac{u(n-2)}{log(n)} \frac{u(n-g-2)}{log(n-g)}, \tag{3.2.7}$$

or equivalently:

$$\pi_g(x) \sim \sum_{2+g\le n\le x} \frac{1}{log(n)} \frac{1}{log(n-g)}. \tag{3.2.8}$$

Assuming the Riemann Hypothesis, the approximation refines to:

$$\pi_g(x) \sim \sum_{2+g\le n\le x} \frac{1}{log(n)} \frac{1}{log(n-g)} + O\left(\sqrt{x-g}\,\frac{log(x)\,log(x-g)}{\sqrt{x}}\right), \quad x \ge 2+g. \tag{3.2.9}$$

Using the Euler-Maclaurin summation formula, the above sum can be approximated by an integral:

$$\pi_g(x) \sim \int_{2+g}^{x} \frac{1}{log(t)} \frac{1}{log(t-g)} dt. \tag{3.2.10}$$

Odd Gaps and Special Cases: The approximation (3.2.10) is not valid for odd gaps g, as all prime numbers except 2 are odd. For an odd gap g, there is at most one pair of primes $(2,2+g)$. Consequently, the prime pair counting function $\pi_g(x)$ simplifies to:

$$\pi_g(x) = \chi_P(2+g)u(x-2-g), \quad g \in 2N+1, \tag{3.2.11}$$

where $u(x)$ is the unit step function. This captures the unique nature of odd gaps, arising from the parity properties of prime numbers.

3.3. Goldbach Prime Pair Counting Function

A central problem in number theory is to construct a function that counts prime pairs satisfying the Goldbach conjecture. The conjecture states that every even integer $2m$ (where $m \in N,\ m \geq 3$) can be expressed as the sum of two prime numbers p and q, with $p, q \in P$.

Indicator Function for Goldbach Prime Pairs: To begin, we define the following indicator function:

$$1_U(n) = 1_P(n)1_P(2m-n), \tag{3.3.1}$$

where $U = \{n \in P\} \cap \{2m-n \in P\}$.

Renaming the prime indicator function $1_P(n)$ as the prime characteristic function $\chi_P(n)$, this becomes:

$$1_U(n) = \chi_P(n)\chi_P(2m-n). \tag{3.3.2}$$

Definition of the Goldbach Prime Pair Counting Function: The Goldbach prime pair counting function, denoted as $G_2(m)$, is defined by the summation:

$$G_2(m) = \sum_{2\leq n} \chi_P\,(n)\chi_P(2m-n). \tag{3.3.3}$$

Using the inequality (3.1.20), we can derive an upper bound for $G_2(m)$:

$$G_2(m) \leq \sum_{2\leq n} \frac{\Lambda(n)}{log(n)} \frac{\Lambda(2m-n)}{log(2m-n)}, \tag{3.3.4}$$

where $\Lambda(n)$ is the von Mangoldt function. For the Goldbach conjecture to hold, the condition $G_2(m) > 0\ \forall m \geq 3$ must be satisfied.

Asymptotic Approximation: From the inequality (3.3.4), the following asymptotic expansion can be derived:

$$G_2(m) \sim \sum_{2\leq n} \frac{\Lambda(n)}{log(n)} \frac{\Lambda(2m-n)}{log(2m-n)}. \tag{3.3.5}$$

Incorporating the asymptotic expression for the prime characteristic function, we deduce:

$$G_2(m) \sim \sum_{2\leq n} \frac{u(n-2)}{log(n)} \frac{u(2m-2-n)}{log(2m-n)}, \tag{3.3.6}$$

or equivalently:

$$G_2(m) \sim \sum_{n=2}^{2m-2} \frac{1}{log(n)} \frac{1}{log(2m-n)}. \tag{3.3.7}$$

Assuming the truth of the Riemann Hypothesis, this expression refines to:

$$G_2(m) \sim \sum_{n=2}^{2m-2} \frac{1}{log(n)} \frac{1}{log(2m-n)} + O\left(\sqrt{2m-2}\,log(\,2m-2)\right). \tag{3.3.8}$$

Integral Approximation: Since the summation diverges as $m \to \infty$, we use the Euler-Maclaurin formula to approximate the sum by an integral:

$$G_2(m) \sim \int_2^{2m-2} \frac{1}{log(t)} \frac{1}{log(2m-t)}\,dt. \tag{3.3.9}$$

3.4. Prime k-Tuple Counting Function

In number theory, a prime k -tuple is a finite collection of prime numbers with constant gaps between consecutive primes. Consider the vector of prime numbers $\vec{p} = (p_1, p_2, \ldots, p_k) \in P^k$, where all primes in the collection satisfy the relationship:

$$p_n - p_1 = g_{n-1}, \quad \forall n = 1,2,\ldots,k, \quad \text{with} g_0 = 0. \tag{3.4.1}$$

This is a generalization of the prime pair counting function $\pi_g(x)$. Using similar techniques, we first construct the indicator function:

$$1_T(n) = \chi_P(n) \prod_{i=1}^{k-1} \chi_P\,(n - g_i), \tag{3.4.2}$$

where:

$$T = \{n \in P\} \cap \bigcap_{i=1}^{k-1} \{n - g_i \in P\}.$$

By induction, this indicator function can be shown to correctly identify prime k -tuples. The prime k -tuple counting function is then defined as:

$$\pi_{(g_1, g_2, \ldots, g_{k-1})}(x) = \sum_{n \le x} \chi_P\,(n) \prod_{i=1}^{k-1} \chi_P\,(n - g_i). \tag{3.4.3}$$

Using induction, we can derive the following inequality:

$$\pi_{(g_1, g_2, \ldots, g_{k-1})}(x) \le \sum_{n \le x} \frac{\Lambda(n)}{log(n)} \prod_{i=1}^{k-1} \frac{\Lambda(n-g_i)}{log(n-g_i)}. \tag{3.4.4}$$

Asymptotic Approximation: From inequality (3.4.4), we obtain the following asymptotic approximation:

$$\pi_{(g_1, g_2, \ldots, g_{k-1})}(x) \sim \sum_{n \le x} \frac{\Lambda(n)}{log(n)} \prod_{i=1}^{k-1} \frac{\Lambda(n-g_i)}{log(n-g_i)}. \tag{3.4.5}$$

If we incorporate the asymptotic approximation of the prime characteristic function, we have:

$$\pi_{(g_1, g_2, \ldots, g_{k-1})}(x) \sim \sum_{n \le x} \frac{u(n-2)}{log(n)} \prod_{i=1}^{k-1} \frac{u(n-g_i-2)}{log(n-g_i)}, \tag{3.4.6}$$

or equivalently:

$$\pi_{(g_1, g_2, \ldots, g_{k-1})}(x) \sim \sum_{2+g_{k-1} \le n \le x} \frac{1}{log(n)} \prod_{i=1}^{k-1} \frac{1}{log(n-g_i)}. \tag{3.4.7}$$

Integral Approximation: Since the above summation diverges as $x \to \infty$, we apply the Euler-Maclaurin formula to approximate it by an integral:

$$\pi_{(g_1, g_2, \ldots, g_{k-1})}(x) \sim \int_{2+g_{k-1}}^{x} \frac{1}{log(t)} \prod_{i=1}^{k-1} \frac{1}{log(t-g_i)}\, dt. \tag{3.4.8}$$

3.5. Prime Gap Function

The prime gap function measures the difference between two successive prime numbers. Formally, it is defined as:

$$g_n = p_{n+1} - p_n = \pi^{-1}(n+1) - \pi^{-1}(n), \tag{3.5.1}$$

where g_n is the prime gap function, p_n is the n -th prime, and $\pi^{-1}(n)$ is the inverse prime counting function.

As discussed in the previous section, the prime gap function is related to the prime counting function $\pi(n)$ via the equation:

$$g_k = \sum_{n=0}^{\infty} \operatorname{sinc}\left(k - \pi(n)\right) \sim \left.\frac{d\pi^{-1}(x+1)}{dx}\right|_{x=k}. \tag{3.5.2}$$

Approximation Using the Prime Number Theorem: According to the prime number theorem, the prime counting function can be approximated as:

$$\pi(n) \sim \mathrm{Li}(n) = \int_2^n \frac{1}{log(t)} dt. \tag{3.5.3}$$

Similarly, the inverse prime counting function is approximated as:

$$\pi^{-1}(n) \sim \mathrm{Li}^{-1}(n). \tag{3.5.4}$$

Thus, the prime gap function can be approximated by:

$$g_k \sim \left.\frac{d\mathrm{Li}^{-1}(x+1)}{dx}\right|_{x=k}. \tag{3.5.5}$$

3.5.1. Asymptotic Approximation

We proceed with a more detailed asymptotic analysis of the prime gap function given by (3.5.5). Starting with:

$$g_k \sim \left.\frac{d\mathrm{Li}^{-1}(x+1)}{dx}\right|_{x=k} \sim \left.\frac{d\mathrm{Li}^{-1}(x)}{dx}\right|_{x=k}, \quad k \to \infty. \tag{3.5.6}$$

Using the derivative of the inverse function, as described in [6], we obtain:

$$g_k \sim \left.\frac{dy}{d\mathrm{Li}(y)}\right|_{y=\mathrm{Li}^{-1}(k)}. \tag{3.5.7}$$

Substituting $d\mathrm{Li}(y)/dy \sim log(\, y)$, we find:

$$g_k \sim log(\, y)|_{y=\mathrm{Li}^{-1}(k)}. \tag{3.5.8}$$

Differential Equation for the Inverse Logarithmic Integral: Consider the differential equation for the inverse logarithmic integral:

$$\frac{d\mathrm{Li}^{-1}(x)}{dx} = log\left(\int \frac{d\mathrm{Li}^{-1}(x)}{dx} dx\right). \tag{3.5.9}$$

Renaming $t = \frac{d\mathrm{Li}^{-1}(x)}{dx}$, the solution to this equation is:

$$\int_a^t \frac{e^u}{u} du = x + c_1. \tag{3.5.10}$$

Since $t: g_k$, we deduce:

$$\int_a^{g_k} \frac{e^u}{u} du \sim k + c_1. \tag{3.5.11}$$

Applying the initial condition:

$$\int_1^{g_k} \frac{e^u}{u} du \sim k, \tag{3.5.12}$$

and assuming the Riemann Hypothesis, combined with the results from [8], the prime gap function can be further approximated as:

$$g_k \sim E^{-1}(k) + O\left(\frac{\log^{5/2}(k)}{\sqrt{k}}\right), \tag{3.5.13}$$

where $E^{-1}(k)$ is the inverse of the integral defined in (3.5.12).

4. Applications of Inversion Using Conformal Mapping

4.1. The von Mangoldt Function

To compute the von Mangoldt function $\Lambda(k)$, we use the integral representation:

$$\Lambda(k) = \frac{1}{2\pi i}\oint_C \frac{k^{log(z)}}{z} D\,(\Lambda, log(\,z))dz, \tag{4.1.1}$$

where $D(\Lambda, log(\,z))$ is the Dirichlet series of the von Mangoldt function, given by:

$$D(\Lambda, log(\,z)) = -\frac{\zeta'(log(z))}{\zeta(log(z))}. \tag{4.1.2}$$

As detailed in [4], this series can be expressed as:

$$-\frac{\zeta'(log(z))}{\zeta(log(z))} = \frac{1}{log(z)-1} - \sum_{\zeta(\rho)=0} \frac{1}{log(z)-\rho}, \tag{4.1.3}$$

where ρ denotes the zeros of the zeta function in the critical strip, including the trivial zeros $\rho = -2m,\ m \in N$.

Contour Integration and Mapping: To evaluate the integral, we consider the branch cut contour at $[0,2\pi)$ using the keyhole contour C, illustrated in Figure **1**.

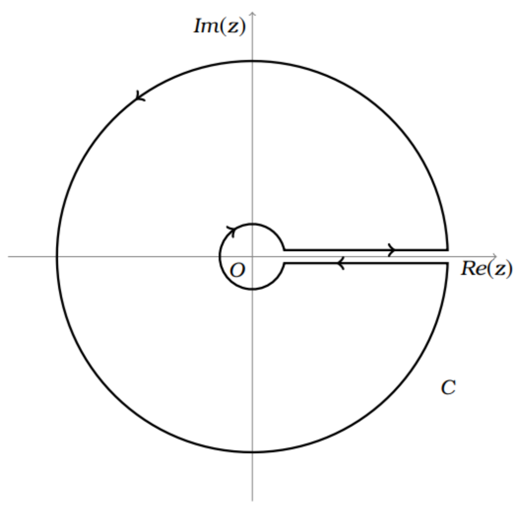

Figure 1: Keyhole contour for evaluating $\Lambda(k)$.

The exponential function maps the real numbers to the interval $(0,+\infty)$. Consequently: - The pole at $log(\,z) = 1$ and the trivial zeros are mapped to this line, which the contour C excludes. - Nontrivial zeros, having an imaginary part, are mapped to circles inside C.

The imaginary parts of the nontrivial zeros lie in the set D, defined as:

$$D = \{w \in R/2\pi Z : \zeta(\rho) = 0,\ 0 < \Re(\rho) < 1,\ w = \Im(\rho) + 2\pi m,\ m \in Z,\ 0 \le w < 2\pi\}. \tag{4.1.4}$$

For brevity, we use the fractional part notation:

$$w = 2\pi \left\{\frac{\Im(\rho)}{2\pi}\right\}, \tag{4.1.5}$$

where $\{\cdot\}$ denotes the fractional part.

Using the residue theorem [6], the integral evaluates as:

$$\Lambda(k) = -\frac{1}{2\pi i} \sum_{\substack{\zeta(\rho)=0 \\ 0<\Re(\rho)<1}} \oint_C \frac{k^{\log(z)}}{z} \frac{1}{\log(z)-\rho} dz, \tag{4.1.6}$$

which simplifies to:

$$\Lambda(k) = -\sum_{\substack{\zeta(\rho)=0 \\ 0<\Re(\rho)<1}} k^{\Re(\rho)+iw}. \tag{4.1.7}$$

4.2. Prime Characteristic Function

Using conformal mapping, the prime characteristic function is expressed as:

$$\chi_P(n) = \frac{1}{log(n)} \frac{1}{2\pi i} \oint_C \frac{F(n; log(z))}{z} \left(-\frac{\zeta'(log(z))}{\zeta(log(z))}\right) dz. \tag{4.2.1}$$

Applying the same techniques as for $\Lambda(k)$, we deduce:

$$\chi_P(n) = -\frac{1}{\log(n)} \sum_{\substack{\zeta(\rho)=0 \\ 0<\Re(\rho)<1}} F(n; \Re(\rho)+iw). \tag{4.2.2}$$

4.3. Mobius Function

Similarly, for the Mobius function $\mu(k)$, we use the integral:

$$\mu(k) = \frac{1}{2\pi i} \oint_C \frac{k^{log(z)}}{z} D(\mu, log(z)) dz, \tag{4.3.1}$$

where the Dirichlet series of the Mobius function is:

$$D(\mu, log(z)) = \frac{1}{\zeta(log(z))}. \tag{4.3.2}$$

The integral evaluates as:

$$\mu(k) = \sum_{\substack{\zeta(\rho)=0 \\ 0<\Re(\rho)<1}} \frac{k^{\Re(\rho)+iw}}{\zeta'(\rho)}. \tag{4.3.3}$$

5. Proof of Errors for Prime Functions

5.1. Prime Characteristic Function

According to [13], the error in the prime counting function $\pi(x)$ compared to the logarithmic integral $\mathrm{li}(x)$, under the assumption of the Riemann Hypothesis, is bounded by:

$$|\pi(x) - \mathrm{li}(x)| \leq \sqrt{x} \frac{log\, x}{8\pi}. \tag{5.1.1}$$

Equivalently, this can be expressed using big-O notation as:

$$\pi(x) = \mathrm{li}(x) + O\left(\sqrt{x}\, log\, x\right). \tag{5.1.2}$$

As $x \to \infty$, applying the average order theorem yields the following estimate:

$$\frac{\pi(x)}{x} = \frac{\mathrm{li}(x)}{x} + O\left(\frac{log\, x}{\sqrt{x}}\right). \tag{5.1.3}$$

Since $\frac{\pi(n)}{n} \sim \chi_P(n)$ and $\frac{\mathrm{li}(n)}{n} \sim \frac{1}{log\, n}$, it follows that:

$$\chi_P(n) = \frac{1}{log\, n} + O\left(\frac{log\, n}{\sqrt{n}}\right), \quad n \to \infty. \tag{5.1.4}$$

5.2. Prime Pair Counting Function

The prime pair counting function $\pi_g(x)$ is defined as:

$$\pi_g(x) = \sum_{n \le x} \chi_P\,(n)\chi_P(n-g). \tag{5.2.1}$$

Using the approximation for the prime characteristic function, this becomes:

$$\pi_g(x) = \sum_{2+g \le n \le x} \left(\frac{1}{log\, n} + O\left(\frac{log\, n}{\sqrt{n}}\right)\right)\left(\frac{1}{log(n-g)} + O\left(\frac{log(n-g)}{\sqrt{n-g}}\right)\right). \tag{5.2.2}$$

Expanding the terms and simplifying, we have:

$$\pi_g(x) = \sum_{2+g \le n \le x} \frac{1}{log\, n\, log(n-g)} + \sum_{2+g \le n \le x} O\left(\frac{log\, n}{\sqrt{n}}\right) O\left(\frac{log(n-g)}{\sqrt{n-g}}\right). \tag{5.2.3}$$

This simplifies further to:

$$\pi_g(x) = \sum_{2+g \le n \le x} \frac{1}{log\, n\, log(n-g)} + (x-2-g) O\left(\frac{log\, x}{\sqrt{x}} \frac{log(x-g)}{\sqrt{x-g}}\right). \tag{5.2.4}$$

Using the properties of big-O notation, we arrive at:

$$\pi_g(x) = \sum_{2+g \le n \le x} \frac{1}{log\, n\, log(n-g)} + O\left(\sqrt{x-g}\, \frac{log\, x\, log(x-g)}{\sqrt{x}}\right). \tag{5.2.5}$$

5.3. Goldbach Prime Pair Function

The Goldbach prime pair function $G_2(m)$ is given by:

$$G_2(m) = \sum_{2 \le n} \chi_P(n)\chi_P(2m-n). \tag{5.3.1}$$

Using the approximation for the prime characteristic function, we have:

$$G_2(m) = \sum_{n=2}^{2m-2} \left(\frac{1}{log\, n} + O\left(\frac{log\, n}{\sqrt{n}}\right)\right)\left(\frac{1}{log(2m-n)} + O\left(\frac{log(2m-n)}{\sqrt{2m-n}}\right)\right). \tag{5.3.2}$$

After expanding and simplifying, this becomes:

$$G_2(m) = \sum_{n=2}^{2m-2} \frac{1}{log\, n\, log(2m-n)} + \sum_{n=2}^{2m-2} O\left(\frac{log\, n}{\sqrt{n}}\right) O\left(\frac{log(2m-n)}{\sqrt{2m-n}}\right). \tag{5.3.3}$$

Simplifying further:

$$G_2(m) = \sum_{n=2}^{2m-2} \frac{1}{log\, n\, log(2m-n)} + (2m-4) O\left(\frac{log(2m-2)\, log\, 2}{\sqrt{2m-2}\sqrt{2}}\right). \tag{5.3.4}$$

Using the properties of big-O notation, the result is:

$$G_2(m) \sim \sum_{n=2}^{2m-2} \frac{1}{\log n \log(2m-n)} + O\left(\sqrt{2m-2} \log(2m-2)\right). \tag{5.3.5}$$

5.4. Prime Gap Function

As shown in [8], the n -th prime p_n satisfies:

$$p_n = \mathrm{Li}^{-1}(n) + O\left(\sqrt{n} \log^{5/2}(n)\right), \tag{5.4.1}$$

where Li^{-1} is the inverse logarithmic integral. Applying the average order theorem, we obtain:

$$\frac{p_n}{n} = \frac{\mathrm{Li}^{-1}(n)}{n} + O\left(\frac{\log^{5/2}(n)}{\sqrt{n}}\right). \tag{5.4.2}$$

Since $\frac{p_n}{n} \sim g_n$ and $\frac{\mathrm{Li}^{-1}(n)}{n} \sim E^{-1}(n)$, it follows that:

$$g_n = E^{-1}(n) + O\left(\frac{\log^{5/2}(n)}{\sqrt{n}}\right), \quad n \to \infty. \tag{5.4.3}$$

6. Conclusions

In this paper, we have presented systematic methods for inverting Dirichlet series and deriving asymptotic approximations for the associated integral kernels. These techniques provide valuable tools for analyzing a wide range of arithmetic functions, particularly those connected to prime numbers, divisors, and related indicators.

We began by investigating inversion techniques using both integral kernels and conformal mapping, providing general formulations and detailed derivations. The asymptotic behavior of the integral kernels was established using methods such as the Riemann-Lebesgue lemma and properties of the sinc function. These approximations allowed us to simplify the inversion process significantly as the parameters approached infinity.

Using the Average Order Theorem as a foundation, we demonstrated numerous applications of these results to classical problems in analytic number theory. Specifically, we applied the methods to approximate key arithmetic functions, including:

- The prime characteristic function $\chi_P(n)$ and its connection to the Prime Number Theorem.
- The prime gap function g_k, which measures the gaps between consecutive primes.
- The prime factor counting function $\omega(n)$, whose behavior aligns with the Hardy-Ramanujan theorem.
- The divisor characteristic function $1_{m|n}$ and divisor counting function $d(n)$, including their asymptotic estimates.
- The greatest common divisor indicator function $1_{gcd(k,n)=m}$ and its generalizations.

Furthermore, we derived approximations for the inverse prime counting function $\pi^{-1}(k)$, connecting it to prime gaps and Stieltjes integrals involving the prime counting function $\pi(x)$. These results highlight the utility of Dirichlet series inversion in addressing problems involving primes and other arithmetic functions.

The techniques developed in this work provide a robust framework for approximating arithmetic functions and analyzing their asymptotic properties. Future research can explore extensions of these methods to other classes of series, such as Mellin or Fourier series, and their applications to broader areas of analytic number theory. Additionally, improving the error bounds of the approximations derived here remains a promising direction for further investigation.

Final Remarks

The inversion of Dirichlet series, combined with asymptotic kernel approximations, bridges a gap between classical analytic methods and modern computational approaches. By providing concrete applications to prime number theory, this work demonstrates the enduring significance of Dirichlet series as a powerful tool in understanding the structure and distribution of primes and other number-theoretic functions.

Conflict of Interest

The authors don't have any conflict of interest

Funding

The author(s) declare that no funding was received for the preparation or publication of this article.

Acknowledgement

The authors would like to thank Nefeli Papadopoulou and Lina Matiadou for their valuable assistance and useful advice on the structure of the paper.

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