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Functional Integration on Constrained Function Spaces I: Foundations

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ABSTRACT

Analogy with Bayesian inference is used to formulate constraints within a scheme for functional integration proposed by Cartier and DeWitt-Morette. According to the analogy, functional counterparts of conditional and conjugate probability distributions are introduced for integrators. The analysis leads to some new functional integration tools and methods that can be applied to the study of constrained dynamical systems.

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1. Introduction

Kinematical constraints (e.g. constraints in the form of boundary conditions) on dynamical systems modeled by differential equations have been well-studied. Typical solution methods are often based on elementary techniques that rely on simple boundary value matching. But subtleties can arise from complicated geometries/topologies, and it becomes necessary to extend the elementary methods — especially in quantum physics. For example, one extension to general geometries makes use of the generalized Green's theorem: By formulating the solution of a differential equation in terms of Green's functions, arbitrary boundary geometries with certain regularity conditions can be treated. Such extensions deal directly with function spaces and the mathematical complexities and subtleties inherent in them.

Meanwhile, dynamically constrained systems (e.g. systems with local symmetries) and their quantization have been — and continue to be — extensively studied for obvious reasons. Solution methods for this constraint type are usually anything but elementary. The vast literature on this topic supports the contention that, here also, the function spaces of the dynamical variables (as opposed to their target manifolds) are of primary importance.

Crucially, from a function space perspective, the distinction between kinematical constraints and dynamical constraints is unnecessary. Both types can be formulated by posing a restricted (or constrained) function space: In practice, the restriction is often imposed indirectly on a target manifold, and it leads to some kind of set-reduction in some appropriate general function space. Consequently, one can anticipate that function spaces furnish a fruitful arena in which to formulate and study *all* constraints.

Moreover, it has long been recognized that functional integration offers reliable if not always acceptably rigorous methods to probe function spaces that model physical systems, so it is not surprising that functional integrals have become useful analytical and numerical tools to study complex, constrained dynamical systems. Accordingly, they offer a means to incorporate and study both kinematical and dynamical constraints under one shared framework.

There are many references in the physics literature that study constraints in functional integrals; largely utilizing formal/heuristic or time-slicing approaches.¹ For a sample, see [1-6]: On the mathematics side, see [7,8] and references therein. The aim of this article is not to supplant those methods — they are certainly useful tools — but to propose a mathematical basis for functional integration on constrained function spaces. The basis is suggested by analogy to Bayesian inference theory, and it affords some guiding principles. With guiding principles in place, useful integration techniques can be developed and tested.

This work will utilize the Cartier/DeWitt-Morette (CDM) scheme as the mathematical foundation of functional integration (without constraints). A short summary is given in appendix 5, but the reader is encouraged to consult [9-11] for background and details. Their approach is similar to, but generalizes, the framework of Albeverio/Høegh-Krohn [12]. Roughly stated, the CDM scheme uses Fourier duality to define linear integral operators on Banach spaces in terms of bona fide measures on Polish spaces. The as-defined functional integrals, which can be characterized by associated *integrators*, then inherit useful properties through their duality relationship. These properties can then be used to reliably manipulate the functional integrals.²

Application of the CDM scheme to spatially unbounded quantum mechanical (QM) systems is well-understood, but how it works under general constraints seems to require new principles. We start by presenting several well-known examples that contain clues to the underlying principles. To begin with, they suggest that constraints add

¹ Formal/heuristic and time-slicing methods are not without merit: Since functional integrals often localize to Riemann-Stieltjes integrals, the standard manipulations *usually* lead to a correct result.

² Since the CDM scheme is restricted to function spaces whose elements are pointed paths that take their values in some manifold \mathbb{X} , i.e. maps $x : [t_a, t_b] \subset \mathbb{R} \rightarrow \mathbb{X}$ with a fixed point $x(t_a) = x_a \in \mathbb{X}$, the functional integrals in this article are strictly *path integrals*. However, the CDM scheme can be extended to include more general function spaces (see e.g. [13]) and the guiding principles we identify are not particular to path integrals in this restricted sense. So the term functional integral will be used interchangeably with path integral.

non-dynamical degrees of freedom, and this requires an enlarged function space. Next, the Bayesian analogy suggests the notions of *conditional integrators* and *conjugate integrators*. Together with the functional counterpart of the Fubini theorem, these tools enable us to construct and manipulate functional integral representations of constrained dynamical systems within the CDM scheme.

This is the main idea of the paper: Constrained dynamical systems require a general state space comprising dynamical *and* non-dynamical degrees of freedom. Specifying particular constraints induces a subset of the general state space that we designate as a constrained function space. The dynamics of any given system is ostensibly represented by an associated functional integral based on an appropriate integrator and constrained function space. Equivalently and more conveniently, the integral over the constrained function space can be replaced by a functional integral over the full general state space but characterized by some conditional integrator. The conditional integrator is defined by functional integral analogs of marginal and conditional probability densities, which in turn are related using the Bayesian inference analogy.

The essence of the main idea is essentially a generalization of the familiar technique often employed to study systems with constraints especially due to symmetry: The physical state/phase space — which respects the system constraints — is replaced by a state/phase space that ‘forgets’ the constraints, and the system constraints are implemented through appropriate functionals formulated in terms of some target manifold. The value of our approach of encoding constraints within conditional integrators in the CDM scheme lies in a shift of perspective from the target manifold to the function space; which affords a probability interpretation along with its guiding intuitions.

Although the primary focus of the paper is a proposed construction of constrained functional integrals, there are some secondary results obtained along the way that we should point out: 1) The Gaussian integrators in the CDM scheme are redefined to include a boundary form and a parameter that encodes a mean path. The new definitions are more useful in the context of constraints with their concomitant sufficient statistics. 2) The complex counterpart of the new Gaussian integrator is likewise defined. Although we do not pursue the idea here, it appears that the complex Gaussian might contain important information regarding the Schrödinger \leftrightarrow diffusion correspondence. Specifically, it might explain when analytic continuation succeeds or fails in this context. 3) In [14] a new integrator within the CDM scheme was introduced based on analogy to a gamma probability distribution. Its utility for incorporating boundary conditions in path integrals was recognized, but its meaning and origin were unclear. Here we learn that the gamma integrator is a natural consequence of the Bayesian analogy. Moreover, the gamma integrator possesses a complex parameter that, when restricted to the natural numbers, reduces to what can be characterized as a Poisson integrator. In consequence, the ‘propagator’ for a dynamical system characterized by a gamma integrator yields an equivalent construction of the Poisson functional integral introduced in [9].

(A caveat; for convenience all variables are assumed unit-less by appropriate normalization.)

2. Motivating Examples

In this section we briefly look at some well-known functional integrals of certain constrained dynamical systems derived using standard semi-heuristic arguments. Our exposition will be rather loose and formal but expedient. The exercise is useful as it gives hints about how to do constrained functional integration in general. We will revisit these systems and their constrained functional spaces in more detail in a companion paper [19] after developing a firmer mathematical basis in this paper.

2.1. Localization

The first class of examples — constrained Feynman path integrals — can be characterized heuristically by the presence of a delta function(al) in the integrand of a path integral. Some particularly prevalent early examples of this type in quantum mechanics were point-to-point transition amplitudes, fixed energy transition amplitudes, and the propagator for a particle on the circle \mathbb{S}^1 . Let’s see what these look like in the CDM scheme for the simplest case of free particles.

In the CDM scheme, the domain of integration for a Gaussian path integral is a space of pointed paths X_a (see appendix 6 — especially for notation). So point-to-point transition amplitudes $K(x_a, x_b)$ are obtained by a suitable delta function in the integrand that ‘pins down’ the loose end of the paths. Standard manipulations [9] reveal that the path integral representing point-to-point transition amplitudes for quadratic actions can be expressed in terms of a restricted domain of paths

$$\begin{aligned} K(x_a, x_b) &= \int_{X_{a,b}} e^{2\pi i \langle x', x \rangle} \mathcal{D}\omega^{(a,b)}(x) = \int_{X_a} \delta(x(t_b), x_b) e^{2\pi i \langle x', x \rangle} \mathcal{D}\omega(x) \\ &= \frac{e^{-\pi i W^{(a,b)}(x')}}{\sqrt{\det[i\hat{G}(t_b, t_b)]}} \end{aligned} \quad (2.1)$$

where $\hat{G}(t_b, t_b)$ is the covariance associated with the gaussian integrator $\mathcal{D}\omega(x)$ defined on the space of pointed paths X_a and $W^{(a,b)}$ is the variance associated with the *restricted* space of point-to-point paths $X_{a,b}$.

Aside from the action phase factor and the resulting normalization³

$$\int_{X_{a,b}} \mathcal{D}\omega^{(a,b)}(x) = \frac{1}{\sqrt{\det[i\hat{G}(t_b, t_b)]}} \quad (2.2)$$

the new gaussian integrator $\mathcal{D}\omega^{(a,b)}(x)$, which is defined on $X_{a,b}$, is characterized by a *different* covariance $\hat{G}^{(a,b)}(t, s)$ that exhibits the same boundary conditions as paths in $X_{a,b}$.

Now consider the other two examples. At the classical level, constraints such as fixed energy and paths on \mathbb{S}^1 can be imposed by means of Lagrange multipliers in the classical action. It is then a standard heuristic argument that the Lagrange multiplier constitutes a non-dynamical, path-independent degree of freedom in the path integral that can therefore be integrated out. Essentially, this introduces what can be characterized as a Dirac delta functional. However, to give rigorous meaning to a delta functional, one would need a theory of distributions on X_a .

An alternative route is to define a Dirac integrator $\mathcal{D}\delta(x)$ that does the duty of a delta functional. It can be thought of as the limit of a Gaussian integrator with vanishing variance, i.e. $|W(x')| \rightarrow 0$. The Dirac integrator reproduces the expected behavior;

$$\int_{X_a} F(x) \mathcal{D}\delta(x) = F(0) \quad (2.3)$$

and

$$\int_{X_a} F(x) \mathcal{D}\delta(M(x)) = \sum_{x_0} \frac{F(x_0)}{\text{Det}M'(x_0)} \quad (2.4)$$

where $M: X_a \rightarrow X_a$ and $M(x_0) = 0$. Otherwise stated, it enforces a localization in the functional integral domain X_a onto the kernel of M .

Similarly, an inverse Dirac integrator $\mathcal{D}\delta^{-1}(x)$ can be formally defined that corresponds to the case $|W(x')| \rightarrow \infty$ so that

$$\int_{X_a} e^{-2\pi i \langle x', x \rangle} \mathcal{D}\delta^{-1}(x) = \delta(x') \quad (2.5)$$

This integrator enforces a localization in the dual space X'_a . In contrast to a Gaussian or Dirac integrator, this type of integrator however is not translation invariant: Given $\langle x' - x'_0, x \rangle = \langle x', x + x_0 \rangle$, which defines x'_0 , we have

³ This particular normalization is fixed a priori from the choice $\int_x \mathcal{D}\omega(x) = 1$.

$$\begin{aligned} \delta(x') &= \int_{X_a} e^{-2\pi i \langle x', x+x_0 \rangle} \mathcal{D}\delta^{-1}(x+x_0) = \int_{X_a} e^{-2\pi i \langle x'-x'_0, x \rangle} \mathcal{D}\delta^{-1}(x+x_0) \\ &\Downarrow \\ e^{2\pi i \langle x'_0, x \rangle} \mathcal{D}\delta^{-1}(x+x_0) &\sim \mathcal{D}\delta^{-1}(x) . \end{aligned} \quad (2.6)$$

Equivalently,

$$\int_{X_a} e^{-2\pi i \langle x', x \rangle} \mathcal{D}\delta^{-1}(x+x_0) = \delta(x' - x'_0) . \quad (2.7)$$

The salient features to note from these three constrained path integral examples are: i) constraints are related to a localization in function space (or its dual), ii) constraints are related to a change in covariance and/or mean, and iii) in general the normalization of a constrained integrator is different than the unconstrained integrator.

2.2. Quotient Spaces

When the target space \mathbb{X} of the pointed paths $x: [t_a, t_b] \rightarrow \mathbb{X}$ can be represented as the base space of a principal fiber bundle $\pi_{\mathbb{G}}: \mathbb{P} \rightarrow \mathbb{X}$, *equivariant* forms on \mathbb{X} can be expressed in terms of associated forms on \mathbb{P} . This technique is essentially symmetry based and allows, for example, determination of propagators on multiply connected spaces, orbifolds, compact Lie groups, and homogeneous spaces.

The principal fiber bundle construction is essentially a generalization of the well-known method of images. In practice, the group structure of the principal fiber bundle allows the space of paths X_a on the base space to be related to a group decomposition of the space of paths P_a on the principal bundle where $p: [t_a, t_b] \rightarrow \mathbb{P}$ with $p(t_a) = p_a$. In this way, paths taking their values in configuration spaces with non-trivial topology and/or geometry can be treated as paths taking their values in the covering space. This renders a simplified function space — to the extent allowed by the covering space. In terms of path integrals, the method can be roughly expressed as

$$\int_{X_a} F(x) \mathcal{D}\omega(x) = \int_G \int_{P_a} \tilde{F}(p \cdot g) \mathcal{D}\tilde{\omega}(p) \mathcal{D}g . \quad (2.8)$$

where G is the space of pointed paths $g: [t_a, t_b] \rightarrow \mathbb{G}$ with $g(t_a) = g_a$ and \mathbb{G} the group manifold.

But invariably the functions of interest are equivariant and covariantly constant which means $p_a \in \mathbb{P}$ is parallel transported. This ‘constraint’ induces a map $g \mapsto h \in \mathbb{H}_{(p_a)}$ into the holonomy group with reference point p_a , and the integral reduces to the standard result

$$\int_{X_a} F(x) \mathcal{D}\omega(x) = \int_{\mathbb{H}_{(p_a)}} \int_{P_a} \tilde{F}(p \cdot h) \mathcal{D}\tilde{\omega}(p) dh . \quad (2.9)$$

The point to be made here is that the two function spaces P_a and X_a are related through a constraint enforced by an integration (and/or summation for multiply connected or discrete holonomy groups). Insofar as finite-dimensional integrals are localized functional integrals, we could loosely say that introducing a constrained integrator on $P_a \times G$ (implicitly) renders the functional integral on the constrained space X_a .

2.3. Discontinuous Spaces

Our final class of examples is comprised of configuration spaces in which $x(t)$ experiences some kind of discontinuity. Particular cases include bounded configuration spaces, barrier penetration, and tunneling. The previous two classes of examples gave only a vague hint of how constraints influence a path integral. However, this third class of examples yields valuable clues and insights.

If we believe that a variational principle lies at the heart of the quantum→classical reduction, then we should re-examine the variational problem in the context of constraints. Consider a boundary in configuration space. For point-to-boundary paths, the correct formulation is a variational problem from a fixed initial point to a manifold in the dependent-independent variable space. This type of variational problem introduces a variable end-point that

can be interpreted as a non-dynamical dependent variable that encodes the implicit constraints imposed by the configuration space discontinuities and boundaries.

To formulate the variational principle for paths taking their values in a space \mathbb{X} that intersects a boundary, consider the $\dim(n + 1)$ dependent-independent variable space $\mathbb{N} = \mathbb{X} \times \mathbb{R}$ with a terminal manifold of dimension $(n + 1) - k$ defined by some set of equations $\{S_k(x, t) = 0\}$ where $k \leq n$, $x \in \mathbb{X}$, and $t \in [t_a, t_b] \subset \mathbb{R}$. Let

$$I(x) = \int_{t_a}^{t_b} f(t, x, \dot{x}) dt$$

be the functional to be analyzed. The extrema of $I(x)$ solve the variational problem for point-to-boundary paths. In particular, for the case of $\mathbb{X} = \mathbb{R}^n$, the variational problem is solved by the usual Euler equations supplemented by 'transversality' conditions (see e.g. [15]).

There are two limiting cases of particular interest. When the terminal manifold in \mathbb{N} coincides with a boundary (or surface) in \mathbb{X} , then $k = 1$ and the transversality conditions reduce to

$$f(t_b, x(t_b), \dot{x}(t_b)) = -\nu \nabla S(x(t_b)) \cdot \dot{x}(t_b) \quad (2.10)$$

where $\nu \neq 0$ is a constant and $x(t_b)$ is on the boundary. For free motion, (2.10) implies that critical paths intersect the boundary transversely.

The other case of interest is when the manifold in the dependent-independent space is 'horizontal', i.e. $x(t_b)$ is fixed and the terminal manifold is a line along the t direction. This clearly corresponds to a point-to-point transition between two fixed points contained in a bounded region. The terminal manifold is determined by $k = n$ equations and the transversality conditions yield

$$f(t_b, x(t_b), \dot{x}(t_b)) = \nabla_{\dot{x}} f(t_b, x(t_b), \dot{x}(t_b)) \cdot \dot{x}(t_b) \quad (2.11)$$

where \dot{x} is a unit vector in the $\dot{x}(t_b)$ direction. If, in particular, $f = L + E$ where L is the Lagrangian of an isolated physical system and E is a constant energy, then this is just the fixed energy constraint $(\partial L / \partial \dot{x}^i) \dot{x}^i - L = E$. Consequently, the variational problem in this case is solved by paths with both end-points fixed that have fixed energy [15].

There are two lessons to learn from this: i) when boundaries are present, we will need to introduce a non-dynamical degree of freedom, and ii) the boundaries will alter certain expectation values of the paths according to the transversality conditions.

At this point, the nature of the new degree of freedom is obscure. However, if one wants a functional integral to represent the solution of a second order partial differential equation with non-trivial boundary conditions, then a consistent formulation emerges if one is willing to associate the new degree of freedom with a non-Gaussian integrator. It turns out that the new integrator is closely related to a gamma probability distribution in the same way that the Gaussian integrator is related to a Gaussian probability distribution.

The nagging question is, Why a gamma integrator? The examples have furnished some clues: not surprisingly they point to probability theory. If the answer can be understood, perhaps formulations of path integral representations of more general differential equations will become evident.

3. Constraints as Conditionals

Consider a physical system with dynamical, topological, and/or geometrical constraints. Postulate that such constraints introduce non-dynamical degrees of freedom. The obvious idea to incorporate these degrees of freedom in a functional integral context is to enlarge the function space. Consequently, construct $B \equiv X \times C$ a Banach product space. The space X corresponds to what is usually thought of as the space of maps, and C will be a space of non-dynamical degrees of freedom induced by any constraints. In a probability context, this additional product structure

would introduce conditional and marginal distributions. In our context, we expect analogous structures; about which little can be said until the nature of the integrators on \mathcal{C} are understood.

3.1. Probability Analogy

Here it is fruitful to develop an analogy with Bayesian inference theory.⁴ Momentarily pretend that B is a probability space. Let $\Theta_X(x)$ and $\Theta_C(c)$ be the marginal probability distributions on X and \mathcal{C} respectively. Bayes' theorem implies

$$\begin{aligned}\widetilde{\Theta}_X(x|c) &= \frac{\widetilde{\Theta}_C(c|x)\Theta_X(x)}{\int_X \widetilde{\Theta}_C(c|x)\Theta_X(x)dx} \\ &=: \mathcal{C}(c|x)\Theta_X(x)\end{aligned}\tag{3.1}$$

where $\widetilde{\Theta}_X$ and $\widetilde{\Theta}_C$ are conditional probability distributions on B . This yields insight into the constraint induced normalization noticed in the examples of the previous section.

This induced normalization is not surprising, because a constraint could alternatively be formulated as a map $M: X \rightarrow Y$ where $y \in Y$ automatically obeys the constraint. Then change of variable techniques in the CDM scheme can be used to show the two associated integrators are related through a functional determinant which is essentially $\mathcal{C}(c|x)$. This is standard, but it shows that the probability interpretation is consistent (at least with change of variables) and it lends credence to the analogy.

So far, we have only made use of Bayes' theorem. To profit further from the analogy, consider an optical setup where plane monochromatic waves are focused onto an observation screen. We wish to study the nature of the light source by placing various non-conducting apertures between the source and the observation screen. Sooner or later we discover that under mild intensities the irradiance pattern on the observation screen is determined by the mean and covariance of the transmittance at each point in the aperture. Moreover, by changing the wavelength and/or intensity of the source, the resulting irradiance pattern can be predicted.

The Bayesian inferential interpretation of these findings is that the *conditional* probability density or *likelihood* $\widetilde{\Theta}_X(x|c)$ — which describes the irradiance pattern for a given aperture — can be factored as a product of a functional $F(x)$ of the transmittance x and a conditional likelihood that only depends on the mean and covariance of the transmittance. In general, the statement is there exist *sufficient statistics* $S_s(x)$ such that

$$\widetilde{\Theta}_X(x|c) = F(x)\widetilde{\Theta}_{S_s(X)}(S_s(x)|c)\tag{3.2}$$

where $F(x)$ is a functional on X and $\widetilde{\Theta}_{S_s(X)}(S_s(x)|c)$ is a likelihood on $S_s(X) \times \mathcal{C}$. In other words, the irradiance pattern only depends conditionally on a (rather special) subset of X . An equivalent statement by way of Bayes' theorem is that the conditional probability density $\widetilde{\Theta}_C(c|x) \propto \Theta_C(c)\widetilde{\Theta}_{S_s(X)}(S_s(x)|c)$ is a functional of sufficient statistics.

There are two key points⁵ illuminated by the analogy. The first point is the effects of a constraint can be inferred from a subset $S_s(X) \subset X$ given $\widetilde{\Theta}_{S_s(X)}(S_s(x)|c)$ and the marginal probability density $\Theta_C(c)$. And the second is the marginal and conditional probability distributions on \mathcal{C} belong to the same conjugate family, i.e.

$$\widetilde{\Theta}_C(c|x) \propto \Theta_C(c)\widetilde{\Theta}_{S_s(X)}(S_s(x)|c).\tag{3.3}$$

There is great value in these two key points: We can understand a constrained system through the constraint distribution and a subset of its dynamical variables, the sufficient statistics. Moreover, given a particular likelihood

⁴ A rather dated but classic reference for the probability concepts introduced in this subsection is [16]; especially Ch. 2.

⁵ These points assume the system is not driven 'too hard' so that the probability distribution that characterizes the system does not change during trials or observations.

function and a set of sufficient statistics, the possible conjugate distributions on C are quite limited. In fact, consulting a table of conjugate priors for standard distributions, one can readily find the associated conjugate families.

There are, no doubt, further lessons to be learned about constraints from the probability correspondence, but at this point we leave the analogy and return to the CDM scheme of functional integration.

3.2. Conditional and Conjugate integrators

Return to B a Banach space of pointed paths, and amend the CDM scheme with the definition⁶

Definition 3.1 Let $B \equiv X \times Y$ be a Banach product space, and let each component Banach space be endowed with CDM scheme data. Define

$$\Theta_{X|Y}(x|y, x'|y') := \frac{\Theta_B(b, b')}{\Theta_Y(y, y')} = \frac{\Theta_B(b, b')}{\int_X \Theta_B(b, b') \mathcal{D}_{\Theta_X, Z_X} x} \quad (3.4)$$

and

$$Z_{X'|Y'}(x'|y') := \frac{Z_{B'}(b')}{Z_{Y'}(y')} = \frac{Z_{B'}(b')}{\int_{X'} Z_{B'}(b') d\mu_{X'}(x')} . \quad (3.5)$$

These two functionals define an associated conditional integrator by

$$\int_B \Theta_{X|Y}(x|y, x'|y') \mathcal{D}_{\Theta_{X|Y}, Z_{X|Y}} x|y := Z_{X'|Y'}(x'|y') . \quad (3.6)$$

The space $\mathbf{F}_{X|Y}(B)$ of constrained integrable functionals consists of functionals defined by

$$F_\mu(x|y) := \int_{B'} \Theta_{X|Y}(x|y, x'|y') d\mu(x'|y') = \int_{B'} \frac{\Theta_B(b, b')}{\Theta_Y(y, y')} d\mu(x'|y') \quad (3.7)$$

where $\mu(x'|y')$ is a conditional measure⁷ on B' . Then the linear integral operator on $\mathbf{F}_{X|Y}(B)$ is given by

$$\int_B F_\mu(x|y) \mathcal{D}_{\Theta_{X|Y}, Z_{X|Y}} x|y = \int_{B'} Z_{X'|Y'}(x'|y') d\mu(x'|y') . \quad (3.8)$$

Proposition 3.1

$$\begin{aligned} \int_B \Theta_{X|Y}(x|y, x'|y') \mathcal{D}_{\Theta_{X|Y}, Z_{X|Y}} x|y &= \frac{1}{Z_{Y'}(y')} \int_B \Theta_B(b, b') \mathcal{D}_{\Theta_B, Z_B} b \\ \Downarrow \\ \mathcal{D}_{\Theta_{X|Y}, Z_{X|Y}} x|y &: \frac{\Theta_Y(y, \cdot)}{Z_{Y'}(\cdot)} \mathcal{D}_{\Theta_B, Z_B} b \end{aligned} \quad (3.9)$$

In particular, since the integrator relation holds for any $y' \in Y'$,

$$\int_B F_\mu(x|y) \mathcal{D}_{\Theta_{X|Y}, Z_{X|Y}} x|y = \frac{1}{Z_{Y'}(y')} \int_B F_\mu(x|y) \Theta_Y(y, y') \mathcal{D}_{\Theta_B, Z_B} b \quad (3.10)$$

⁶ The use of Θ in both the probability and functional integral context is meant to be suggestive, but it should be kept in mind that the same symbol is referring to two distinct objects that should not be confused.

⁷ The conditional measure is well defined as the restriction of μ to the appropriate sub- σ -algebra over B' .

most often with $\langle y', y \rangle = S_s(y)$ or $\langle y', y \rangle = 0$ for all $y \in Y$.

Proof. The proof follows immediately from definition 3.1 and the relevant CDM definitions. \square

Evidently expressing integrals like $\int_B F(b) \mathcal{D}b$ when B is a product space requires knowledge of the ‘marginal’ and ‘conditional’ integrators on the component spaces. Of course, when elements in X and Y are independent, the conditional integrator on B reduces to a simple product of standard integrators on X and Y . But we anticipate that constraints induce a dependence between elements in X and Y .

Now specialize to the case when Y represents non-dynamical degrees of freedom — ostensibly due to constraints. As suggested by the optical diffraction example, postulate that the physical system is described by ‘sufficient statistics’⁸ and that Θ_Y and $\Theta_{Y|X}$ belong to the same conjugate family. Then knowledge of the ‘likelihood’ functional $\Theta_{X|Y}(x|y, x'|y')$ implies knowledge of the conjugate family of $\Theta_Y(y, y')$ and vice versa. Consequently, the heuristic integral $\int_B F(b) \mathcal{D}b$ will be well defined in terms of known functionals.

Accordingly, the Bayesian analogy suggests the definition:

Definition 3.2 Conjugate integrators are characterized by

$$\Theta_{Y|X}(y|x, \cdot) \propto \Theta_{S_s(X)|Y}(S_s(x)|y, \cdot) \Theta_Y(y, \cdot) \quad (3.11)$$

where

$$\int_Y \Theta_Y(y, y') \mathcal{D}_{\Theta_Y, Z_Y} y = Z_Y(y') \quad (3.12)$$

and the proportionality is fixed by normalization.

Note that this implies (by analogy with Bayes’ theorem)

$$\Theta_{X|Y}(x|y, \cdot) \propto \Theta_B((S_s(y), x), \cdot) . \quad (3.13)$$

This property suggests the solution strategy

$$\begin{aligned} \int_{\tilde{X}} F_\mu(\tilde{x}) \mathcal{D}_{\Theta_{\tilde{X}}, Z_{\tilde{X}}} \tilde{x} &:= \int_B F_\mu(x|y) \mathcal{D}_{\Theta_{X|Y}, Z_{X|Y}} x|y \\ &:= \int_B \tilde{F}_\mu(S_s(y), x, \cdot) \Theta_Y(y, \cdot) \mathcal{D}_{\Theta_B, Z_B} b \\ &:= \int_X \left[\int_Y \tilde{F}_\mu(S_s(y), x, \cdot) \Theta_Y(y, \cdot) \mathcal{D}_{\Theta_Y, Z_Y} y \right] \mathcal{D}_{\Theta_X, Z_X} x \\ &:= \int_X \tilde{G}_\mu(x) \mathcal{D}_{\Theta_X, Z_X} x \end{aligned} \quad (3.14)$$

where the integral on the left is interpreted as a constrained functional integral, i.e. an integral over the constrained function space \tilde{X} , the third line employs functional Fubini (Prop. A.3 in [14]), and \tilde{G}_μ can be interpreted as a constrained functional depending on the constraints only through sufficient statistics. This is the functional integral analog of (3.1).

Notice that the statement is equally valid with $X \leftrightarrow Y$ if one knows some $S_s(X)$; hence suggesting an alternative solution strategy

⁸ In a functional integral context, ‘sufficient statistics’ is interpreted naturally as a localization in the space of paths precipitated by some constraint. For a QM system, constraints restrict the domain of the evolution operator, and it is fruitful to identify ‘sufficient statistics’ with the evolution operator’s spectra at $t = t_a$. In effect, the conditional integrator loosely represents a spectral measure.

$$\begin{aligned}
\int_{\tilde{X}} F_{\mu}(\tilde{x}) \mathcal{D}_{\Theta_{\tilde{X}}, Z_{\tilde{X}}} \tilde{x} &:= \int_B F_{\mu}(y|x) \mathcal{D}_{\Theta_{Y|X}, Z_{Y|X}} y|x \\
&:= \int_B \tilde{F}_{\mu}(S_s(x), y, \cdot) \Theta_X(x, \cdot) \mathcal{D}_{\Theta_B, Z_B} b \\
&:= \int_Y \left[\int_X \tilde{F}_{\mu}(S_s(x), y, \cdot) \Theta_X(x, \cdot) \mathcal{D}_{\Theta_X, Z_X} x \right] \mathcal{D}_{\Theta_Y, Z_Y} y \\
&:= \int_Y \tilde{H}_{\mu}(y) \mathcal{D}_{\Theta_Y, Z_Y} y .
\end{aligned} \tag{3.15}$$

Both strategies can be fruitfully employed depending on one's knowledge of a system's relevant sufficient statistics.

The important point worth emphasizing is that $\Theta_{Y|X}$ and Θ_Y belong to the same family of integrators, and they are simply related through the sufficient statistics that describe the integrator on X . This quickly narrows the search for an integrator associated with a particular constraint.

4. Conclusion

We used Bayesian inference theory within the CDM scheme for functional integration to propose a basis for formulating constrained functional integrals. The probability analogy introduces two main ideas. The first idea is that a constrained dynamical system is partially characterized by a subset of its associated function space — the functional analog of sufficient statistics. (Quite often the subset will be a finite-dimensional subspace isomorphic to some target manifold associated with a physical model.) The second idea is that a functional integral whose domain is a constrained function space can instead be constructed on an enlarged function space equipped with conditional and conjugate integrators.

However natural the probability analogy may seem, the usefulness of the defined functional integrals rests on their efficacy — which in turn depends on establishing physically relevant conditional and conjugate integrators. To this end, we describe in detail four particularly pertinent integrator families in appendix 6 and use them in the companion paper [19] to re-examine the motivating examples of §2 in light of our proposed formulation.

Appendix

5. CDM Scheme

The Cartier/DeWitt-Morette scheme [9-11] defines functional integrals in terms of the data $(B, \Theta, Z, \mathbf{F}(B))$.

Here B is a separable (usually) infinite dimensional Banach space with norm $\|b\|$ where $b \in B$ is an $L^{2,1}$ map $b: [t_a, t_b] \in \mathbb{R} \rightarrow \mathbb{C}^m$. The dual Banach space $B' \ni b'$ is a space of linear forms such that $\langle b', b \rangle_B \in \mathbb{C}$ with an induced norm given by

$$\|b'\| = \sup_{b \neq 0} |\langle b', b \rangle| / \|b\|.$$

Assume B' is separable. Then B' is Polish and consequently admits complex Borel measures μ .

Θ and Z are bounded, μ -integrable functionals $\Theta: B \times B' \rightarrow \mathbb{C}$ and $Z: B' \rightarrow \mathbb{C}$. The functional $\Theta(b, \cdot)$ can be thought of as the functional analog of a probability distribution function and $Z(b')$ the associated characteristic functional.

The final datum is the space of integrable functionals $\mathbf{F}(B)$ consisting of functionals $F_{\mu}: B \rightarrow \mathbb{C}$ defined relative to μ by

$$F_{\mu}(b) := \int_{B'} \Theta(b, b') d\mu(b') . \tag{5.1}$$

If $\mu \mapsto F_{\mu}$ is injective, then $\mathbf{F}(B)$ is a Banach space endowed with a norm $\|F_{\mu}\|$ defined to be the total variation of μ .

These data are used to define an integrator $D_{\Theta,Z}b$ on B by

$$\int_B \Theta(b, b') \mathcal{D}_{\Theta,Z}b = Z(b'). \quad (5.2)$$

This defines an integral operator $\int_B \mathcal{D}_{\Theta,Z}b$ on the Banach space $\mathbf{F}(B)$;

$$\int_B F_\mu(b) \mathcal{D}_{\Theta,Z}b = \int_B Z(b') d\mu(b') =: \int_B F_\mu(b) \mathcal{D}_{\Theta,Z}(b + b_o) \quad (5.3)$$

for some fixed $b_o \in B$.⁹ The integral operator is a bounded linear form on $\mathbf{F}(B)$ with

$$\left| \int_B F_\mu(b) \mathcal{D}_{\Theta,Z}b \right| \leq \|F_\mu\|. \quad (5.4)$$

6. Integrator Families

6.1. Gaussian Family

Before defining Gaussian integrators we establish some terminology. Let X_{ab} be the space of $L^{2,1}$ pointed paths $x: [t_a, t_b] \subseteq \mathbb{R} \rightarrow \mathbb{X}$ such that $x(t_a) =: x_a \in \mathbb{X}$ with \mathbb{X} a real, *flat* differentiable manifold and boundary condition $c_B(x(t_b), \dot{x}(t_b)) \in \mathbb{R}$. The variance $W: X_{ab}' \times X_{ab}' \rightarrow \mathbb{C}$ is a bilinear form with domain $D_W = X_{ab}'$ defined by

$$W(x_1', x_2') =: \frac{1}{2} \{ \langle x_1', Gx_2' \rangle + \langle x_2', Gx_1' \rangle \} =: \langle x', Gx' \rangle_{\{1,2\}} \quad (6.1)$$

where the covariance $G: X_{ab}' \rightarrow X_{ab}$ is non-negative definite. Associated with the variance is a *symmetric, closed*¹⁰ form $Q: X_{ab} \times X_{ab} \rightarrow \mathbb{C}$;

$$Q(x_1, x_2) = \langle Dx, x \rangle_{\{1,2\}} - B(x_1, x_2) \quad (6.2)$$

where $B(x_1, x_2)$ is a symmetric boundary form¹¹ and $D: X_{ab} \rightarrow X_{ab}'$ is a linear map. Let $\{\bar{x}\}$ be the set of “mean” paths endowed with suitable boundary conditions defined to be the stationary points of the effective action Γ associated with the variance W ; that is $\delta\Gamma(x)/\delta x(t)|_{x=\bar{x}} = -2\pi i \delta \langle x', x \rangle / \delta x'(t)|_{x=\bar{x}} = -2\pi i x_{\bar{x}}' \equiv 0$ with boundary conditions $\bar{x}(t_a) = x_a$ and $c_B(\bar{x}(t_b), \dot{\bar{x}}(t_b))$. Note that $Q(\bar{x}) = \Gamma(\bar{x})$, because for quadratic Q the action and effective action coincide so $\bar{x} = x_{cr}$ where $Dx_{cr} = 0$ is a critical path of Q .

Let $X_{\hat{x}_{cr}}$ be the space $X_{00} \setminus \text{Ker}(D)$ where the paths in X_{00} have vanishing boundary conditions $x(t_a) = 0$ and $c_B(x(t_b), \dot{x}(t_b)) = 0$. Then, *restricting to this factor space*, we have $DG = \text{Id}_{X_{\hat{x}_{cr}}'}$ and $GD = \text{Id}_{X_{\hat{x}_{cr}}}$ and so $W(x')$ and $Q(x)$ are inverse *up to a boundary form* in this case. Further, any $x \in X_{00}$ can be reached from a given x_{cr} by $x = x_{cr} + Gx'$ for all $x' \in X_{\hat{x}_{cr}}'$. Consequently, each non-trivial critical/“mean” path spawns a copy of $X_{\hat{x}_{cr}}$ in X_{00} .¹²

Definition 6.1 A family of Gaussian integrators $\mathcal{D}\omega_{\bar{x},Q}(x)$ is characterized by¹³

⁹ There are ways to motivate translation invariance of the integrator, but here we will simply define it that way.

¹⁰ Q closed means that its domain D_Q can be endowed with a Hilbert space structure. It can be shown that for Q symmetric and closed there exists a unique self-adjoint operator $A: D_Q \rightarrow D_Q$ such that $D_A \subset D_Q$ and $Q(x_1, x_2) = \langle x_1, Ax_2 \rangle$ for any $x_1 \in D_Q$ and $x_2 \in D_A$ ([17], Th. 4.6.8). The boundary form $B(x_1, x_2)$ enforces $D_A = D_Q$.

¹¹ For example, if $Q(x_1, x_2) = \int \dot{x}_1 \dot{x}_2 dt$, then $D = d^2/dt^2$ and $B(x_1, x_2) = 1/2(x_1 \dot{x}_2|_{t_a}^{t_b} + \dot{x}_1 x_2|_{t_a}^{t_b})$. So $B(x) = x \dot{x}|_{t_a}^{t_b} \neq 0$ unless $x(t_a) = \dot{x}(t_b) = 0$ or $x(t_b) = \dot{x}(t_a) = 0$. As a less trivial check, the reader can verify that $B(\bar{x}) = -\omega[(x_a^2 + x_b^2) \cos \omega(t_b - t_a) - 2x_a x_b] / \sin \omega(t_b - t_a) = Q(\bar{x})$ when $D = d^2/dt^2 + \omega^2$ on the space of paths with both endpoints fixed.

¹² This brief characterization of W and Q can and should be rigorously developed in the context of linear operators on the Hilbert space associated with a constrained function space \bar{X} . In particular, one should apply results regarding self-adjoint extensions of D and their associated spectra in this context. A thorough study would produce a useful translation dictionary between the rigorous mathematics describing linear operators on Hilbert spaces and their Gaussian functional integral counterparts.

¹³ This definition uses a different normalization from the usual Gaussian integrator in the CDM scheme. Both definitions are valid: we choose this normalization because it seems more consistent with definitions of other integrator families and it highlights the role of the functional determinant.

$$\begin{aligned}\Theta_{\bar{x},Q}(x, x') &= e^{2\pi i \langle x', x \rangle - (\pi/s)Q(x-\bar{x})} \\ Z_{\bar{x},W}(x') &= \sqrt{s} \text{Det}(W)^{1/2} e^{2\pi i \langle x', \bar{x} \rangle - \pi s W(x')} \end{aligned} \quad (6.3)$$

where $\langle x', x \rangle \in \mathbb{R}$, $s \in \{1, i\}$, and the functional determinant is assumed to be well-defined.

The Gaussian integrator family is defined in terms of the primitive integrator $\mathcal{D}x$ by

$$\mathcal{D}\omega_{\bar{x},Q}(x) := e^{-(\pi/s)Q(x-\bar{x})} \mathcal{D}x \quad (6.4)$$

where $\mathcal{D}x$ is characterized by

$$\Theta_{0,\text{Id}}(x, x') = \exp\{2\pi i \langle x', x \rangle - (\pi/s)\text{Id}(x)\} ; \quad Z_{0,\text{Id}}(x') = \sqrt{s} e^{-\pi s \text{Id}(x')} . \quad (6.5)$$

Loosely, the primitive integrator $\mathcal{D}x$ (which is characterized by zero mean and trivial covariance) is the infinite dimensional analog of the Lebesgue measure on \mathbb{R}^n . Note that W (and hence $\text{Det}W$), inherits the boundary conditions imposed on x , and note the normalizations

$$\int_{X_{00}} \mathcal{D}\omega_{0,\text{Id}}(x) = \int_{X_{00}} e^{-(\pi/s)\text{Id}(x)} \mathcal{D}x = \sqrt{s} \quad (6.6)$$

and

$$\int_{X_{ab}} \mathcal{D}\omega_{\bar{x},Q}(x) := \int_{x_{cr}} \int_{X_{\bar{x}_{cr}}} \mathcal{D}\omega_{\bar{x},Q}(x) = \int_{\bar{x}} \sqrt{s} \text{Det}(W)^{1/2} . \quad (6.7)$$

Three points to emphasize: The fiducial Gaussian integrator $\mathcal{D}\omega_{0,\text{Id}}(x)$ is associated with the bona fide Banach space X_{00} where paths vanish on the boundary and the primitive integrator is translation invariant, i.e. $\mathcal{D}(x_1 - x_2) = \mathcal{D}(x_1)$. For any given $x_{cr} = \bar{x}$, the middle integral in (6.7) can therefore be written as an integral over X_{00} by a change of integration variable $x - \bar{x} \mapsto \tilde{x}$ with $\tilde{x}(t_{a,b}) = 0$ since the primitive integrator is translation invariant. Finally, since there is a copy of $X_{\bar{x}_{cr}}$ for each non-trivial zero mode, we see that the integral over the full space X_{ab} must include a sum/integral over all $x_{cr} = \bar{x}$.

The resemblance between the functional form of $Z_{\bar{x},W}(x')$ and the exponential multiplying the primitive integrator motivates the standard practice in quantum field theory of defining the effective action functional. First, note that

$$\mathcal{Z}_{\bar{x},W}(x') := \frac{Z_{\bar{x},W}(x')}{Z_{\bar{x},W}(0)} = e^{2\pi i \langle x', \bar{x} \rangle - \pi s W(x')} =: e^{(\pi/s)\Gamma_{\bar{x}}'(x')} \quad (6.8)$$

is the characteristic functional of the normalized integrator $\frac{1}{Z_{0W}(0)} \mathcal{D}\omega_{\bar{x},Q}(x)$. Moreover,

$$\begin{aligned} \frac{1}{2\pi i} \frac{\delta}{\delta x'(t)} \ln \mathcal{Z}_{\bar{x},W}(x') &= \left(\bar{x}(t) - \frac{s}{2i} \frac{\delta W(x')}{\delta x'(t)} \right) \\ &= \frac{1}{Z_{\bar{x},W}(x')} \frac{\delta}{\delta x'(t)} \int_{X_{\bar{x}_{cr}}} e^{2\pi i \langle x', x \rangle} \mathcal{D}\omega_{\bar{x},Q}(x) \\ &=: \mathcal{X}_{x'}(t) . \end{aligned} \quad (6.9)$$

From the second equality it follows that $\mathcal{X}_{x'}: [t_a, t_b] \rightarrow \mathbb{X}$ is an $L^{2,1}$ function with $\mathcal{X}_{x'}(t_a) = x_a$ and $\dot{\mathcal{X}}_{x'}(t_b) = \dot{x}_b$ for all $x' \in \mathcal{X}'_{\bar{x}_{cr}}$; also $\mathcal{X}_{x'}(t) = \bar{x}(t)$ where x'_{cr} is a critical point of W . So define the effective action $\Gamma_{x'}$ evaluated at $\mathcal{X}_{x'} \in X_{\bar{x}_{cr}}$ by

$$(\pi/s)\Gamma_{x'}(\mathcal{X}_{x'}) := -(\pi/s)\Gamma_{x'}'(x') = \pi s W(x') - 2\pi i \langle x', \mathcal{X}_{x'} \rangle . \quad (6.10)$$

It follows that

$$\frac{i}{2s} \frac{\delta \Gamma_{x'}(\mathcal{X}_{x'})}{\delta \mathcal{X}_{x'}(t)} = x'(t) \quad (6.11)$$

which means

$$\Gamma_{x'}(\mathcal{X}_{x'}) = \int_{t_a}^{t_b} \frac{\delta \Gamma_{x'}}{\delta \mathcal{X}_{x'}(t)} \mathcal{X}_{x'}(t) dt - \frac{1}{4} \int_{t_a}^{t_b} \int_{t_a}^{t_b} \frac{\delta \Gamma_{x'}}{\delta \mathcal{X}_{x'}(t')} G(t', t) \frac{\delta \Gamma_{x'}}{\delta \mathcal{X}_{x'}(t)} dt' dt. \quad (6.12)$$

Essentially, the sum of the exponentiated effective action over all zero modes is the Fourier transform of $\frac{1}{Z_{0,W(0)}} \mathcal{D}\omega_{\mathcal{X}_{x'}, Q}(x)$. Explicitly,

$$\begin{aligned} \mathcal{Z}_{\mathcal{X}_{x'}, W}(x') &= \frac{1}{Z_{0,W(0)}} \int_{\mathcal{X}_{cr}=\mathcal{X}_0} e^{2\pi i \langle x', \mathcal{X}_{x'} \rangle} \int_{X_{00}} e^{2\pi i \langle x', x \rangle - (\pi/s) Q(x)} \mathcal{D}x \\ &= \int_{\mathcal{X}_{cr}=\mathcal{X}_0} e^{2\pi i \langle x', \mathcal{X}_{x'} \rangle} Z_{0,W}(x') \\ &= \int_{\mathcal{X}_{cr}=\mathcal{X}_0} e^{-(\pi/s) \Gamma_{x'}(\mathcal{X}_{x'})}. \end{aligned} \quad (6.13)$$

Notice that, since Q is quadratic, the functional integral is easily evaluated, and the effective action at $x' = x'_{cr}$ is trivially $\Gamma_{x'}(\mathcal{X}_{x'})|_{x'=x'_{cr}} = Q(\bar{x})$.

Gaussian integrators can be readily generalized to non-Gaussian integrators based on non-quadratic action functionals $S: X_{ab} \times X_{ab} \rightarrow \mathbb{C}$ in the CDM scheme — in which case the effective action becomes a useful tool. Now we have $\tilde{Z}_{\bar{x}, S}(x') := \int_{X_a} e^{2\pi i \langle x', x \rangle} \mathcal{D}\omega_{\bar{x}, S}(x)$,

$$\tilde{Z}_{\bar{x}, S}(x') := \frac{\tilde{Z}_{\bar{x}, S}(x')}{Z_{0,S(0)}} =: e^{(\pi/s) \tilde{\Gamma}'_{\bar{x}}(x')} =: e^{2\pi i \langle x', \bar{x} \rangle - \pi s W_S(x')}, \quad (6.14)$$

and

$$(\pi/s) \tilde{\Gamma}'_{\bar{x}}(\mathcal{X}_{x'}) := (\pi s) W_S(x') - 2\pi i \langle x', \mathcal{X}_{x'} \rangle. \quad (6.15)$$

However, $\tilde{\Gamma}'_{\bar{x}}(\mathcal{X}_{x'})|_{x'=x'_{cr}} \neq S(\bar{x})$ and W_S is no longer a simple bilinear form so the effective action for $x' \neq x'_{cr}$ is more complicated. We still get

$$\tilde{Z}_{\mathcal{X}_{x'}, S}(x') = \int_{\mathcal{X}_{cr}} e^{2\pi i \langle x', \mathcal{X}_{x'} \rangle} \mathcal{Z}_{0,S}(x') = \int_{\mathcal{X}_{cr}} e^{-(\pi/s) \tilde{\Gamma}'_{\bar{x}}(\mathcal{X}_{x'})} \quad (6.16)$$

where now x_{cr} is a critical point of $S(x)$. But, as is well known, $\tilde{Z}_{0,S}(x')$ is generically not computable in closed form.

To see how conditional Gaussian integrators work, form the product space $X_{ab} \times Y_{ab}$. Suppose a Gaussian integrator on $X_{ab} \times Y_{ab}$ is characterized by a positive definite quadratic form \tilde{Q} with mean \tilde{m} and *vanishing boundary term*. Put $\tilde{m} = (\tilde{x}, \tilde{y})$ and

$$\tilde{G} = \begin{pmatrix} G_{xx} & G_{xy} \\ G_{yx} & G_{yy} \end{pmatrix}. \quad (6.17)$$

Then¹⁴

$$\tilde{Q}((x, y) - \tilde{m}) = Q_X(x - \tilde{m}_{x|y}) + Q_Y(y - \tilde{y}) \quad (6.18)$$

where $Q_Y(y_1, y_2) = \langle D_{yy} y_1, y_2 \rangle$,

¹⁴ It can be shown that Q_X and Q_Y are positive definite since \tilde{G} is positive definite.

$$Q_X(x_1, x_2) = \langle (G_{xx} - G_{xy}D_{yy}G_{yx})^{-1}x_1, x_2 \rangle \quad (6.19)$$

and

$$\tilde{m}_{x|y} = \tilde{x} + G_{xy}D_{yy}(y - \tilde{y}) . \quad (6.20)$$

So the Gaussian integrator on $X_{ab} \times Y_{ab}$ is

$$\mathcal{D}\omega_{\tilde{m}, \tilde{Q}}(x, y) = e^{-(\pi/s)\tilde{Q}((x,y)-\tilde{m})}\mathcal{D}(x, y) . \quad (6.21)$$

On the other hand,

$$\mathcal{D}\omega_{\tilde{y}, Q_Y}(y) = e^{-(\pi/s)Q_Y(y-\tilde{y})}\mathcal{D}y . \quad (6.22)$$

Therefore, the conditional Gaussian integrator is

$$\mathcal{D}\omega_{\tilde{m}_{x|y}, Q_{X|Y}}(x|y) = e^{-(\pi/s)Q_X(x-\tilde{m}_{x|y})}\mathcal{D}(x, y) \quad (6.23)$$

which yields

$$\int_{X_{ab}} \mathcal{D}\omega_{\tilde{m}_{x|y}, Q_{X|Y}}(x|y) = \sum_{\tilde{m}_{x|y}} \frac{\text{Det}(Q_X+Q_Y)^{-1/2}}{\text{Det}(Q_Y)^{-1/2}} . \quad (6.24)$$

In particular, let $M: X_0 \rightarrow Y_0$ be a homeomorphism such that $Q_1 = Q_2 \circ M$. If $Y_0 = X_0$ then $G_{xy} = G_{yx} = 0$ since the x are independent Gaussian variables. Then formally,

$$\mathcal{D}\omega_{0, Q_1}(x) = \frac{\mathcal{D}\omega_{0, Q_1}(x|y)}{\mathcal{D}\omega_{0, Q_2}(y|x)} \mathcal{D}\omega_{0, Q_2}(y) . \quad (6.25)$$

But

$$\frac{\mathcal{D}\omega_{0, Q_1}(x|y)}{\mathcal{D}\omega_{0, Q_2}(y|x)} \sim \frac{\text{Det}(Q_1)^{-1/2}}{\text{Det}(Q_2)^{-1/2}} \quad (6.26)$$

so we get the standard result for a change of covariance;

$$\int_{X_0} e^{-(\pi/s)Q_2(x)} \mathcal{D}_1 x = \frac{\text{Det}(Q_2)^{-1/2}}{\text{Det}(Q_1)^{-1/2}} \quad (6.27)$$

where $\mathcal{D}_1 x$ is the primitive integrator on X_0 . Obviously the same condition holds for $1 \leftrightarrow 2$ with $\mathcal{D}_2 x$ the primitive integrator on $M(X_0)$.

6.2. Complex Gaussian Family

The previous subsection took the parameter $s \in \{1, i\}$.¹⁵ This restriction can be lifted by defining a complex Gaussian integrator.

Definition 6.2 Let Z_{ab}^2 be the space of $L^{2,1}$ pointed functions $(z, \underline{z}): [t_a, t_b] \subseteq \mathbb{R} \rightarrow \mathbb{M}^{\mathbb{C}}$ such that $(z, \underline{z})(t_a) =: (z_a, \underline{z}_a) \in \mathbb{M}^{\mathbb{C}}$ and $(z, \underline{z})(t_b) =: (z_b, \underline{z}_b) \in \mathbb{M}^{\mathbb{C}}$ with $\mathbb{M}^{\mathbb{C}}$ a flat complex manifold. A complex Gaussian family of integrators $\mathcal{D}\omega_{w, Q^{\mathbb{C}}}(w)$ on $W_{ab} \equiv Z_{ab}^2$ is characterized by

¹⁵ That Gaussian integrators based on non-negative definite *real* G can be defined for $s \in \{1, i\}$ reflects the validity of the Schrödinger \leftrightarrow diffusion correspondence through analytic continuation. However, analytic continuation does not maintain this correspondence in general. It is natural to conjecture that the analytic continuation Schrödinger \leftrightarrow diffusion correspondence will break down precisely when G_{zz} and/or $G_{\underline{z}\underline{z}}$, defined below, do not vanish.

$$\begin{aligned}\Theta_{\bar{w}, Q^{\mathbb{C}}}(w, w') &= e^{2\pi i \langle w', w \rangle - \pi Q^{\mathbb{C}}(w - \bar{w})} \\ Z_{\bar{w}, W^{\mathbb{C}}}(w') &= \text{Det}(W^{\mathbb{C}})^{1/2} e^{2\pi i \langle w', \bar{w} \rangle - \pi W^{\mathbb{C}}(w')} \end{aligned} \quad (6.28)$$

where $w := (z, \underline{z}) \in W_{ab}$, $w' := (z', \underline{z}') \in W'_{ab}$, and $\langle w', w \rangle \in \mathbb{C}$. The complexified variance $W^{\mathbb{C}}(w'_1, w'_2) = \langle w'_1, G^{\mathbb{C}} w'_2 \rangle$ where the complex covariance matrix $G^{\mathbb{C}}$ has the block form

$$G^{\mathbb{C}} = \begin{pmatrix} G_{zz} & G_{z\underline{z}} \\ G_{z\underline{z}} & G_{\underline{z}\underline{z}} \end{pmatrix}. \quad (6.29)$$

with $\Re(\langle w'_1, G^{\mathbb{C}} w'_2 \rangle) \geq 0$ and $G^{\mathbb{C}}$ not necessarily Hermitian.¹⁶ As in the real case, put

$$\mathcal{D}\omega_{\bar{w}, Q^{\mathbb{C}}}(w) = e^{-\pi Q^{\mathbb{C}}(w - \bar{w})} \mathcal{D}w \quad (6.30)$$

where $\mathcal{D}w$ is characterized by

$$\Theta_{0, \text{Id}}(w, w') = \exp\{2\pi i \langle w', w \rangle - \pi \text{Id}(w)\} ; \quad Z_{0, \text{Id}}(w') = e^{-\pi \text{Id}(w')}. \quad (6.31)$$

At the level of functional integrals, evidently there is little difference between the real and complex Gaussian families. The value in the complex case comes when the domain of integration is localized yielding complex line integrals. We suspect complex Gaussian integrators will find application in open dynamical systems.

6.3. Gamma Family

Definition 6.3 Let T_0 be the space of continuous pointed maps $\tau: (\mathbb{T}_+, t_a) \rightarrow (\mathbb{C}_+, 1)$ where $\mathbb{T}_+ \subseteq \mathbb{R}_+$ and $\mathbb{C}_+ := \mathbb{R}_+ \times i\mathbb{R}$ is the right-half complex plane. T_0 is an abelian group under point-wise multiplication in the first component and point-wise addition in the second. Let β' be a fixed element in the dual group T'_0 of linear characters $\tau': T_0 \rightarrow \mathbb{C}$. A gamma family of integrators $\mathcal{D}\gamma_{\alpha, \beta'}(\tau)$ on T_0 is characterized by¹⁷

$$\begin{aligned}\Theta_{\alpha, \beta'}(\tau, \tau') &= e^{i \langle \tau', \tau \rangle - \langle \beta', \tau \rangle} \tau^\alpha \\ Z_{\alpha, \beta'}(\tau') &= \text{Det}(\beta' - i\tau')^{-\alpha} \end{aligned} \quad (6.32)$$

where $\alpha \in \mathbb{C}$, τ^α is defined point-wise by $\tau^\alpha(t) := e^{\alpha \log \tau(t)}$ with the principal value prescription¹⁸ for $\log \tau(t)$, and the functional determinant $\text{Det}(\beta' - i\tau')$ is assumed to be well-defined.

The integrator family is defined in terms of the primitive integrator $\mathcal{D}\tau$ by

$$\mathcal{D}\gamma_{\alpha, \beta'}(\tau) := e^{-\langle \beta', \tau \rangle} \tau^\alpha \mathcal{D}\tau \quad (6.33)$$

where $\mathcal{D}\tau$ is characterized by

$$\Theta_{0, \text{Id}'}(\tau, \tau') = \exp\{i \langle \tau', \tau \rangle - \langle \text{Id}', \tau \rangle\} ; \quad Z_{0, \text{Id}'}(\tau') = \Gamma(0). \quad (6.34)$$

Whereas the primitive integrator $\mathcal{D}x$ is the infinite dimensional analog of the translation invariant measure on \mathbb{R}^n ; the primitive integrator $\mathcal{D}\tau$, for real $\tau(t)$, is the analog of the scale invariant measure on \mathbb{R}_+ . Note that (6.33) requires $|\langle \beta', \tau \rangle| \geq 0$ in order to be well-defined.

¹⁶ If $\underline{z} = z^*$ then $(G^{\mathbb{C}})^\dagger = G^{\mathbb{C}}$.

¹⁷ This definition is somewhat modified from the original definition in [14]; the old definition was in terms of a particular realization of β' evaluated on a subspace of T_0 rendering it an ordinary function.

¹⁸ So the branch cut lies outside the range of τ .

Experience indicates that a prominent sufficient statistic characterizing gamma-type paths is an upper bound $|\tau(t)| \leq |c|$ for all $t \in [t_a, t_b]$ and for some constant $c \in \mathbb{C}_+$ — much like fixed end-points can characterize the sufficient statistics associated with gaussian paths. The obvious tool to enforce this constraint is the functional analog of Heaviside; yielding a ‘cut-off’ gamma family that generalizes the previous definition but reduces to it as $|c| \rightarrow \infty$.

Definition 6.4 Let T_0 be the space of continuous pointed maps $\tau: (\mathbb{T}_+, t_a) \rightarrow (\mathbb{C}_+, 1)$. Let β' be a fixed element in the dual group T'_0 and fix some fiducial $\tau_0 \in T_0$ such that $\langle \beta', \tau_0 \rangle = c \in \mathbb{C}_+$. A lower gamma family of integrators $\mathcal{D}\gamma_{\alpha, \beta', c}(\tau)$ on T_0 is characterized by

$$\begin{aligned}\Theta_{\alpha, \beta'}(\tau, \tau') &= e^{i\langle \tau', \tau \rangle - \langle \beta', \tau \rangle} \tau^\alpha \\ \mathcal{Z}_{\alpha, \beta', c}(\tau') &= \frac{\gamma(\alpha, c)}{\text{Det}(\beta' - i\tau')^\alpha}\end{aligned}\quad (6.35)$$

where $\gamma(\alpha, c)$ is the lower incomplete gamma function given by

$$\gamma(\alpha, c) = \Gamma(\alpha) e^{-c} \sum_{n=0}^{\infty} \frac{c^{\alpha+n}}{\Gamma(\alpha+n+1)}, \quad (6.36)$$

and the functional determinant $\text{Det}(\beta' - i\tau')$ is assumed to be well-defined.

An upper gamma family of integrators $\mathcal{D}\Gamma_{\alpha, \beta', c}(\tau)$ is defined similarly where

$$\Gamma(\alpha, c) = \Gamma(\alpha) - \gamma(\alpha, c) \quad (6.37)$$

is the upper incomplete gamma function.

Using this notion, the fiducial gamma integrator represented by $\mathcal{D}\gamma_{0, Id', \infty}(\tau)$ (equivalently $\mathcal{D}\Gamma_{0, Id', 0}(\tau)$). It is normalized up to a factor of $\Gamma(0)$;

$$\frac{1}{\Gamma(0)} \int_{T_0} \mathcal{D}\gamma_{0, Id', \infty}(\tau) = 1 = \frac{1}{\Gamma(0)} \int_{T_0} \mathcal{D}\Gamma_{0, Id', 0}(\tau), \quad (6.38)$$

but the other family members yield

$$\frac{1}{\Gamma(\alpha)} \int_{T_0} \mathcal{D}\gamma_{\alpha, \beta', \infty}(\tau) = \text{Det}\beta'^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_{T_0} \mathcal{D}\Gamma_{\alpha, \beta', 0}(\tau). \quad (6.39)$$

Eventually in applications we will run into factors of $\int_{T_0} \mathcal{D}\tau$. Rather than normalizing everything by constantly dividing out this factor, we will *define* it to be $\int_{T_0} \mathcal{D}\tau = 1$. This is consistent with $\lim_{z \rightarrow 0} 1/z^0 = 1$ formally applied to (6.39).

Put $B_a = X_a \times T_0$. For Θ_X Gaussian and Θ_T gamma, use the relation for conjugate integrators to get

$$\begin{aligned}\int_{B_a} \Theta_{X|T}(x|\tau, \cdot) \mathcal{D}_{\Theta_{X|T}, Z_{X|T}} x|\tau &= \int_{B_a} \Theta_{X|T}(x|\tau, \cdot) \frac{\Theta_T(\tau, \cdot)}{Z_T} \mathcal{D}_{\Theta_B, Z_B} b \\ &\propto \int_{B_a} \Theta_{S_s(T)|X}(S_s(\tau)|x, \cdot) \frac{\Theta_X(x, \cdot) \Theta_T(\tau, \cdot)}{Z_T} \mathcal{D}_{\Theta_B, Z_B} b.\end{aligned}$$

This suggests that integrals of conditional functionals on $X_a \times T_0$ be understood as

$$\int_{B_a} F_\mu(x|\tau) \mathcal{D}_{\Theta_{X|T}, Z_{X|T}} x|\tau = \int_{B_a} \tilde{F}_\mu(S_s(\tau), x) \mathcal{D}\omega_{\tilde{x}, Q}(x) \mathcal{D}\gamma_{\alpha, \beta', c}(\tau) \quad (6.40)$$

when $S_s(\tau)$ is a sufficient statistic for the integrator family characterized by Θ_X . This is just a specialization of the solution strategy (3.14), and it plays a prominent role in the solution of differential equations defined on bounded regions.

6.4. Poisson Family

Take the lower gamma integrator and regularize by replacing $\gamma(\alpha, c)$ with the regularized lower incomplete gamma function $P(\alpha, c) := \gamma(\alpha, c)/\Gamma(\alpha)$. Restrict the parameters such that $\alpha = n \in \mathbb{N}$, $\beta' = \lambda Id'$, and $\text{Det}(\beta') = \lambda \text{Det}(Id') := \lambda$.

Note that, for $N \in \text{Pois}(c)$ a Poisson random variable, we have

$$\text{Pr}(N < n) = \sum_{k < n} e^{-c} \frac{c^k}{k!}. \quad (6.41)$$

Hence,

$$\text{Pr}(N \geq n) = \sum_{k=n}^{\infty} e^{-c} \frac{c^k}{k!} = P(n, c) = \frac{1}{\Gamma(n)} \int_{T_0} \mathcal{D}\gamma_{n, Id', c}(\tau) \quad (6.42)$$

which, in particular, implies

$$\frac{1}{\Gamma(0)} \int_{T_0} \mathcal{D}\gamma_{0, Id', c}(\tau) = \sum_{k=0}^{\infty} e^{-c} \frac{c^k}{k!}. \quad (6.43)$$

On the other hand,

$$e^{-c} \frac{c^k}{k!} = \frac{e^{-c}}{k!} \int_0^c \dots \int_0^c d\tau_1, \dots, d\tau_k. \quad (6.44)$$

Not surprisingly, $\text{Pois}(c)$ is closely related to the lower gamma integrator which motivates the following definition:

Definition 6.5 Let T_0 be the space of continuous pointed maps $\tau: (\mathbb{T}_+, t_a) \rightarrow (\mathbb{C}_+, 1)$ endowed with a lower gamma family of integrators. Let $\alpha = n \in \mathbb{N}$ and $\langle \beta', \tau_0 \rangle = c$ with $c \in \mathbb{C}_+$. The Poisson integrator family $\mathcal{D}\pi_{n, \beta', c}(\tau)$ is characterized by

$$\begin{aligned} \Theta_{n, \beta'}(\tau, \tau') &= e^{i\langle \tau', \tau \rangle - \langle \beta', \tau \rangle} \tau^n \\ Z_{n, \beta', c}(\tau') &= \frac{P(n, c)}{\text{Det}(\beta' - i\tau')^n}. \end{aligned} \quad (6.45)$$

The Poisson family is defined in terms of the primitive integrator $\mathcal{D}\tau$ by

$$\mathcal{D}\pi_{n, \beta'}(\tau) := e^{-\langle \beta', \tau \rangle} \tau^n \mathcal{D}\tau. \quad (6.46)$$

Note the normalization of the fiducial Poisson integrator

$$\int_{T_0} \mathcal{D}\pi_{0, \beta', c}(\tau) = 1, \quad (6.47)$$

and the rest of the family

$$\int_{T_0} \mathcal{D}\pi_{n, \beta', c}(\tau) = P(n, c). \quad (6.48)$$

For quantum physics applications, $\Re(\tau(t)) = 0$ so that $\tau: (\mathbb{T}_+, t_a) \rightarrow (i\mathbb{R}, 0)$. In this restricted case T_0 becomes a Banach space over \mathbb{R} , and it is useful to define the 'shifted' Poisson integrator by

$$\mathcal{D}\hat{\pi}_{n, \beta', \tau_0}(\tau) := e^{-\langle \beta', (\tau - \tau_0) \rangle} \tau^n \mathcal{D}\tau. \quad (6.49)$$

Then use the shifted Poisson integrator to define the Poisson expectation of β' with respect to τ_0 ;

$$\langle \beta' \rangle_{\tau_0} := \int_{\tau_0} \mathcal{D}\hat{\pi}_{0,\beta',\tau_0}(\tau) = e^{\langle \beta', \tau_0 \rangle}. \quad (6.50)$$

If we take $\langle \beta', \tau_0 \rangle = i \int_{t_a}^{t_b} \beta(t) dt$, then

$$\begin{aligned} \langle \beta' \rangle_{\tau_0} &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{t_a}^{t_b} \beta'(t_1) \cdots \int_{t_a}^{t_b} \beta'(t_n) dt_1, \dots, dt_n \\ &= \sum_{n=0}^{\infty} \int_{t_a}^{t_b} i\beta'(t_1) \int_{t_1}^{t_b} i\beta'(t_2) \cdots \int_{t_n}^{t_b} i\beta'(t_n) dt_1, \dots, dt_n \end{aligned} \quad (6.51)$$

where $t_a \leq t_1 < \cdots < t_n \leq t_b$. Note that $\frac{\partial}{\partial t_b} \langle \beta' \rangle_{\tau_0} = i\beta'(t_b) \langle \beta' \rangle_{\tau_0}$, so the Poisson expectation solves a first-order evolution equation.

Conflict of Interest

The authors declare that they have no conflict of interest.

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