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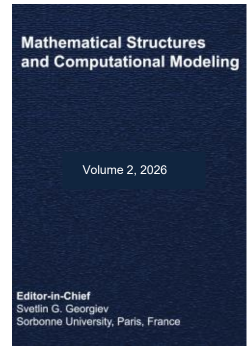

Pizzetti's Formula for Weighed Spherical Mean and its Applications

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ABSTRACT

This article provides a generalization of Pizzetti's formula for weighed spherical mean. This weighed spherical mean is decomposed into a series of Laplace-Bessel operators. As applications, we present expressions for solutions of singular differential equations using Pizzetti's formula for the weighted spherical mean.

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1. Introduction

Pizzetti's formula [1, 2]

$$\int_{S(x,r)} f(y)dS = \sum_{k=0}^{\infty} \frac{r^{2k}}{2^{2k}k!(2k+1)!!} \Delta^k f(x), \tag{1}$$

expresses the average value of a smooth function over a Euclidean sphere $S_n(r)$ of radius r as a power series in the radius r of the sphere, where the coefficients are given by iterations of the Laplace operator Δ.

Formula (1) is widely useful in the theory of partial differential equations (PDEs) and can be used to generate mean-value theorems for solutions to certain differential equations. A generalized mean value theorem, formulated in terms of an arbitrary Borel measure with support in the unit real ball, was established in [3] for solutions of a system of homogeneous partial differential equations.

Pizzetti-type formulas are studied from different points of view [4, 5, 6] and admit generalizations to other geometric structures, including Riemannian manifolds [7], symmetric spaces [8, 9], and H -type groups [10].

In this article, we establish a generalization of Pizzetti's formula for the weighted spherical mean. The weighted spherical mean is expressed as a series involving the powers of Laplace–Bessel operator

$$\Delta_\gamma = \sum_{k=1}^{n} \left(\frac{\partial^2}{\partial x_k^2} + \frac{\gamma_k}{x_k} \frac{\partial}{\partial x_k} \right).$$

As applications, we derive representations for solutions of singular differential equations.

The rest of the paper is organized as follows. In Section 2 Introduces the Laplace-Bessel operator, the relevant function spaces, the notion of weighted generalized functions, the weighted delta-function, and the fundamental solution for Laplace–Bessel operator. Also defines the generalized translation operator and the corresponding convolution. In Section 3 the weighted functional r^λ generated by quadratic form defined and B -harmonic functions (solutions of $\Delta_\gamma u = 0$) discussed. Theorem establishing the explicit form of the fundamental solution of iterated Laplace–Bessel operator was proved in Section 4. Section 5 presents the Taylor–Delsarte series expansion, first in the one-dimensional case and then extended to the multidimensional case. In Section 6 the weighted spherical mean was defined and the main result: a Pizzetti-type formula expanding this mean as a series in powers of the Laplace–Bessel operator was proved. The generalized Pizzetti formula was applied to obtain explicit series representations for solutions of three classes of singular partial differential equations with Laplace–Bessel operator: the B -parabolic (heat) equation, the B -hyperbolic (wave-type) equation, and an B -elliptic (modified Helmholtz) in Section 7.

2. Definitions and Preliminaries

Let R^n be n -dimensional Euclidean space, $x \in R^n$, $\gamma = (\gamma_1, \dots, \gamma_n)$, $\gamma_1 > 0, \dots, \gamma_n > 0$. All functions we will consider on the open orthant $R_+^n = \{x \in R^n, x_1 > 0, \dots, x_n > 0\}$ and on the closed orthant $\overline{R}_+^n = \{x \in R^n, x_1 \geq 0, \dots, x_n \geq 0\}$.

In this article, we study the Laplac-–Bessel operator of the form

$$\Delta_\gamma = \sum_{k=1}^{n} (B_{\gamma_k})_{x_k}, \tag{2}$$

where $(B_{\gamma_k})_{x_k} = \frac{\partial^2}{\partial x_k^2} + \frac{\gamma_k}{x_k} \frac{\partial}{\partial x_k}$ is the Bessel operator, and $k = 1, \dots, n$.

Dealing with the Laplace-Bessel operator (2) we should incorporate power weights into the integration and employ a measure given by $x^\gamma dx$, where $x = (x_1, \dots, x_n) \in R_+^n$, $x^\gamma = x_1^{\gamma_1} \cdot \ldots \cdot x_n^{\gamma_n}$. Using such a power weight we should restrict our observation only by positive variables. However, due to symmetry with respect to the origin, each obtained solution can be evenly extended to negative values of the variables.

Let $\Omega \subset R^n$ be an open set symmetric with respect to each hyperplane $x_i = 0$, $i = 1, \dots, n$. Define $\Omega_+ = \Omega \cap R^n_+$ and $\overline{\Omega}_+ = \Omega \cap \overline{R}^n_+$, so that $\Omega_+ \subseteq R^n_+$ and $\overline{\Omega_+} \subseteq \overline{R}^n_+$.

The notation $C^m(\Omega_+)$ denotes the set of m -times continuously differentiable functions on Ω_+. By $C^m(\overline{\Omega}_+)$ we mean the subset of functions in $C^m(\Omega_+)$ such that all derivatives with respect to x_i ($i = 1, \dots, n$) can be continuously extended to $x_i = 0$.

The class $C^m_{ev}(\overline{\Omega}_+)$ consists of functions $f \in C^m(\overline{\Omega}_+)$ such that $\left.\frac{\partial^{2k+1} f}{\partial x_i^{2k+1}}\right|_{x_i=0} = 0$ for all nonnegative integers k with $2k+1 \le m$ and for all $i = 1, \dots, n$ (see [11], p. 21). Functions in $C^m(\overline{\Omega}_+)$ can be extended evenly to the negative semiaxes by each variable.

Suppose, $C^\infty_{ev}(\overline{\Omega}_+) = \bigcap_{m=0}^\infty C^m_{ev}(\overline{\Omega}_+)$. We set $C^\infty_{ev}(\overline{R}^n_+) = C^\infty_{ev}$.

Let $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+) = D_{ev}(\overline{\Omega}_+)$ be the set of compactly supported functions $f \in C^\infty_{ev}(\overline{\Omega}_+)$.

The space $L^\gamma_p(\Omega_+)$, where $1 \le p < \infty$, consists of functions that are measurable on $\overline{\Omega}_+$ and even with respect to each of their variables x_i for $i = 1, \dots, n$ such that if $f \in L^\gamma_p(\Omega_+)$, then

$$\int_{\Omega_+} |f(x)|^p x^\gamma dx < \infty, \qquad x^\gamma = x_1^{\gamma_1} \cdot \ldots \cdot x_n^{\gamma_n}.$$

We will use notations $L^\gamma_p = L^\gamma_p(R^n_+)$ and

$$||f||_{p,\gamma} = \left(\int_{\mathrm{R}^n_+} |f(x)|^p x^\gamma dx\right)^{1/p}. \tag{3}$$

By $L^\gamma_{p,loc}(\Omega_+)$ we denote the set of functions u defined almost everywhere on $\overline{\Omega}_+$ such that $u\phi \in L^\gamma_p(\Omega_+)$ for all $\phi \in D_{ev}(\overline{\Omega}_+)$. Let $D_{ev}{}'(\overline{\Omega}_+)$ be the dual space to $D_{ev}(\overline{\Omega}_+)$. Each function $u \in L^\gamma_{1,loc}(\Omega_+)$ is associated with a *regular weight generalized function* $u \in D_{ev}{}'(\overline{\Omega}_+)$ acting according to the rule

$$(u, \phi)_\gamma = \int_{\Omega_+} u(x)\phi(x)x^\gamma dx, \quad \phi \in \mathrm{D}_{ev}(\overline{\Omega}_+).$$

All other generalized functions $u \in D'_{ev}(\overline{\Omega}_+)$ will be called *singular weight generalized functions*. We will use the notation $D'_{ev} = D'_{ev}(\overline{R}^n_+)$.

Weighted delta-function $\delta_\gamma \in D'_{ev}$ is a defined by the equality (by analogy with [12] p. 247)

$$(\delta_\gamma, \phi)_\gamma = \phi(0), \quad \phi(x) \in \mathrm{D}_{ev}.$$

Weighted delta-function is a singular weighted generalized function. The fact that this generalized function is weighted explained as follows. Let

$$\omega_\varepsilon(x) = \begin{cases} C_\varepsilon e^{-\frac{\varepsilon^2}{\varepsilon^2 - |x|^2}}, & |x| \le \varepsilon \\ 0 & |x| > \varepsilon, \end{cases}$$

where C_ε is selected such that

$$\int_{\mathrm{R}^n_+} \omega_\varepsilon(x) x^\gamma dx = 1.$$

Since

$$\lim_{\varepsilon \to +0} \int_{\mathrm{R}^n_+} \omega_\varepsilon(x)\varphi(x)x^\gamma dx = \varphi(0), \qquad \varphi \in \mathrm{D}_{ev}$$

we have

$$(\omega_\varepsilon(x), \phi(x))_\gamma \to (\delta_\gamma(x), \phi(x))_\gamma, \qquad \varepsilon \to +0, \qquad \phi \in \mathrm{D}_{ev}.$$

Considering that for convenience we will write

$$(\delta_\gamma, \phi)_\gamma = \int_{\mathrm{R}_+^n} \delta_\gamma(x)\phi(x)x^\gamma dx = \phi(0)$$

and understand it in the sense of limit of delta-shaped sequence.

Part of a ball $|x| \leq r$, $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$ belonging to R_+^n we will denote by $B_r^+(n)$. The boundary of $B_r^+(n)$ denoted by $S_r^+(n)$ consists of a part of a sphere $\{x \in R_+^n : |x| = r\}$ and of parts of coordinate hyperplanes $x_i = 0$, $i = 1, \ldots, n$ such that $|x| \leq r$. The integral by $S_1^+(n)$ with measure $x^\gamma dx$ is given by

$$|S_1^+(n)|_\gamma = \int_{S_1^+(n)} x^\gamma dS = \frac{\prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}. \tag{4}$$

Dealing with Laplace-Bessel operator we define a weighted fundamental solution $E_\gamma = E_\gamma(x)$ as a weighed distribution that solves a linear PDE with a δ_γ as the source term: $LE_\gamma = \delta_\gamma$, $E_\gamma \in D'_{ev}$.

For example, fundamental solution of Δ_γ is a functional from D'_{ev}. Namely, let $x \in R_n^+$, $n > 1$ and

$$E_\gamma(x) = \begin{cases} \frac{1}{|S_1^+(2)|_\gamma} \ln |x|, & n + |\gamma| = 2; \\ \frac{|x|^{2-n-|\gamma|}}{(2-n-|\gamma|)|S_1^+(n)|_\gamma}, & n + |\gamma| > 2; \end{cases}$$

then $B_{\gamma_j} E_\gamma \in L^1_{loc,\gamma}(R_+^n)$ and $\Delta_\gamma E_\gamma = \delta_\gamma$. So $E_\gamma \in D'_{ev}$ is a weighted fundamental solution of the operator Δ_γ.

Consider the generalized translation operator

$$({}^\gamma\mathbf{T}_x^y f)(x) = {}^\gamma\mathbf{T}_x^y f(x) = ({}^{\gamma_1}T_{x_1}^{y_1} \ldots {}^{\gamma_n}T_{x_n}^{y_n} f)(x), \tag{5}$$

on the space of functions that are integrable with the measure $x^\gamma dx$ on R_+^n. In (5) each of one-dimensional generalized translation $\gamma_i T_{x_i}^{y_i}$ acts for $i = 1, \ldots, n$ according to

$$({}^{\gamma_i}T_{x_i}^{y_i} f)(x) = \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma_i}{2}\right)} \times$$

$$\times \int_0^\pi f(x_1, \ldots, x_{i-1}, \sqrt{x_i^2 + \tau_i^2 - 2x_i y_i \cos\varphi_i}, x_{i+1}, \ldots, x_n) \sin^{\gamma_i - 1}\varphi_i \, d\varphi_i. \tag{6}$$

Next we will use notation

$$C(\gamma) = \pi^{-\frac{n}{2}} \prod_{i=1}^n \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}.$$

The generalized convolution generated by the multi-dimensional generalized translation $\gamma \boldsymbol{T}_x^y$ is defined by the formula

$$(f * g)_\gamma(x) = (f * g)_\gamma = \int_{\mathrm{R}_+^n} f(y) \, ({}^\gamma\mathbf{T}_x^y g)(x) y^\gamma dy. \tag{7}$$

The primary purpose of the fundamental solution is to express the solution of a nonhomogeneous equation, providing a way to represent the solution in a concise and meaningful manner.

Let $\Omega_+ \subset R^n_+$ be a bounded set, $f \in D_{ev}$ and vanish outside Ω_+. Then the function u defined by

$$(E_\gamma * f)_\gamma(x) = \int_{\Omega_+} f(y)\, ({}^\gamma\mathbf{T}^y_x E_\gamma)(x) y^\gamma dy$$

satisfies the Poisson equation with Laplace-Bessel operator $\Delta_\gamma u = f$.

3. Weighed Functional r^λ and B-harmonic Functions

When dealing with the Laplace-Bessel operator Δ_γ, we use weighed functionals defined in previous section and corresponding singular functionals.

Let $r = |x| = \sqrt{x_1^2+\ldots+x_n^2}$. For $Re\lambda > -(n+|\gamma|)$ the *weighed functional* r^λ is defined by the formula

$$(r^\lambda, \phi)_\gamma = \int_{\mathrm{R}^n_+} r^\lambda \phi(x) x^\gamma dx, \qquad \phi \in \mathrm{D}_{ev}. \tag{8}$$

For $Re\lambda \leq -(n+|\gamma|)$ we define the functional (8) by the method of analytic continuation with respect to the parameter λ. Functional (8) as a function of λ has simple poles at points

$$\lambda = -(n+|\gamma|), -(n+|\gamma|+2), -(n+|\gamma|+4), \ldots \tag{9}$$

It is easy to see that for $Re\lambda > -(n+|\gamma|)$, the functional (8) is differentiable with respect to the parameter λ:

$$\frac{\partial}{\partial\lambda}(r^\lambda, \phi)_\gamma = \int_{\mathrm{R}^n_+} r^\lambda \ln r\, \phi(x) x^\gamma dx,$$

which implies that the functional r^λ is analytic with respect to the parameter λ in the domain $Re\lambda > -(n+|\gamma|)$.

Homogeneous B -elliptic equation with the Laplace-Bessel operator Δ_γ (see (2))

$$\Delta_\gamma u = 0, \tag{10}$$

is a generalization of the classical elliptic equation. A function that satisfies the differential equation (10) is called B *-harmonic*.

The Laplace-Bessel operator Δ_γ of a function r^λ is given by

$$\Delta_\gamma r^\lambda = \lambda(\lambda+n+|\gamma|-2) r^{\lambda-2}.$$

This formula is critical in mathematical physics, showing that r^λ is B -harmonic in the sense that $\Delta_\gamma r^\lambda = 0$ if $\lambda = 0$ or $\lambda = 2-n-|\gamma|$ for $n+|\gamma| > 2$.

When dealing with the Laplace-Bessel operator, we employ the multidimensional Hankel transform, defined by

$$\mathbf{F}_\gamma[f](\xi) = \mathbf{F}_\gamma[f(x)](\xi) = \hat{f}(\xi) = \int_{\mathrm{R}^n_+} f(x) \mathbf{j}_\gamma(x;\xi) x^\gamma dx, \qquad f \in L_1^\gamma(\mathrm{R}^n_+), \tag{11}$$

where

$$\mathbf{j}_\gamma(x;\xi) = \prod_{i=1}^n j_{\frac{\gamma_i-1}{2}}(x_i\xi_i),\ \gamma_1 > 0, \ldots, \gamma_n > 0,$$

the symbol j_ν is used for the normalized Bessel function of the first kind

$$j_\nu(x) = \frac{2^\nu \Gamma(\nu+1)}{x^\nu} J_\nu(x), \tag{12}$$

J_ν is Bessel function of the first kind. It is clear that $\boldsymbol{F}_\gamma[\delta_\gamma](\xi) = 1$.

Let $f \in L_1^\gamma(R_+^n)$ be of bounded variation with respect to each x_i, $i = 1,\ldots,n$, in a neighborhood of a point x of continuity of f. Then, for $\gamma > 0$, the inversion formula

$$\mathbf{F}_\gamma^{-1}[\hat{f}(\xi)](x) = f(x) = \frac{2^{n-|\gamma|}}{\prod_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \mathbf{F}_\gamma[\hat{f}(\xi)](x) = \frac{2^{n-|\gamma|}}{\prod_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \int_{\mathrm{R}_+^n} \mathbf{j}_\gamma(x,\xi)\, \hat{f}(\xi) \xi^\gamma d\xi$$

holds.

Let $\rho^2 = |\xi|^2 = \sum_{i=1}^n \xi_i^2$. Hankel transform of r_γ^{2m} is

$$\mathbf{F}_\gamma[r^{2m}](\xi) = \mathrm{D}_{n,\gamma}(m) \begin{cases} \rho^{-n-|\gamma|-2m}, & m \neq k, m \neq -\frac{n+|\gamma|}{2} + 2; \\ (-1)^m \Delta_\gamma^m \delta_\gamma, & m = k; \\ \rho^{-n-|\gamma|-2m}\, ln\, \rho, & m = -\frac{n+|\gamma|}{2} + k, \end{cases} \tag{13}$$

where $\delta_\gamma = \delta_\gamma(\xi)$ is weighted delta–function, $k = 0,1,2,\ldots$ and

$$\mathrm{D}_{n,\gamma}(m) = \begin{cases} \dfrac{2|\gamma|+2m \prod_{i=1}^n \Gamma(\frac{\gamma_i+1}{2})\, \Gamma(\frac{n+|\gamma|}{2}+m)}{\Gamma(-m)}, & m \neq k, m \neq -\frac{n+|\gamma|}{2} + 2; \\ 1, & m = k; \\ \prod_{i=1}^n \Gamma(\frac{\gamma_i+1}{2})\, \dfrac{(-1)^{\frac{n+|\gamma|}{2}+m} 2^{|\gamma|+2m+1}}{\left[-\frac{n+|\gamma|}{2}+m\right]!\, \Gamma(-m)}, & m = -\frac{n+|\gamma|}{2} + k \end{cases}$$

4. Iterated Laplace-Bessel Operator

The iterated Laplace operator is used not only in PDEs but also presents a fundamental tool in analysis, geometry, and physics. In this section we consider iterated Laplace-Bessel operator, obtain its fundamental solution and decomposition formula.

The iterated Laplace-Bessel operator plays a significant role in various fields, including partial differential equations with Bessel operator and weighted spherical mean theory. The iterated Laplace-Bessel operator is a linear operator, and its eigenfunctions are the B-polyharmonic functions of finite degree. In [11] a function was said to be "B-polyharmonic" if it was annihilated by some power of the Laplace–Bessel operator. Now we will call such functions "B-polyharmonic of finite degree". Namely, if $u(x) \in C_{ev}^{2p}(\Omega_+)$ and $\Delta_\gamma^p u = 0$ for all $x \in \Omega_+$, then the function $u(x)$ is called *B-polyharmonic of finite degree* p in the open set Ω_+.

When the Laplace-Bessel operator Δ_γ (2) is applied multiple m -times to a function u, it results the iterated Laplace-Bessel operator

$$\Delta_\gamma^m u = \underbrace{\Delta_\gamma(\Delta_\gamma(\ldots(\Delta_\gamma}_{m-\text{times}} u)\ldots)).$$

By multinomial theorem we can write

$$\Delta_\gamma^m u = \left((B_{\gamma_1})_{x_1} + (B_{\gamma_2})_{x_2} + \cdots + (B_{\gamma_n})_{x_n}\right)^m u =$$

$$= \sum_{\substack{k_j \geq 0 \\ |k|=m}} \binom{m}{k_1, k_2, \ldots, k_n} (B_{\gamma_1})_{x_1}^{k_1} (B_{\gamma_1})_{x_2}^{k_2} \ldots (B_{\gamma_n})_{x_n}^{k_n} u,$$

where $|k| = k_1 + k_2 + \ldots + k_n$,

$$\binom{m}{k_1, k_2, \ldots, k_n} = \frac{m!}{k_1! k_2! \ldots k_n!}.$$

Using notations

$$(\mathrm{B}_\gamma)^k u(x) = (B_{\gamma_1})^{k_1}_{x_1}(B_{\gamma_2})^{k_2}_{x_2}\ldots(B_{\gamma_n})^{k_n}_{x_n}u(x_1,\ldots,x_n), \qquad B_{\gamma_i} = (B_{\gamma_i})_{x_i} = \frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$$

and $k! = k_1! \cdot \ldots \cdot k_n!$ we can write

$$\Delta_\gamma^m u = \sum_{|k|=m}\frac{m!}{k!}(\mathrm{B}_\gamma)^k u(x). \tag{14}$$

The next formula holds, with r^λ defined by (8):

$$r^\lambda = \frac{\Delta_\gamma^m r^{\lambda+2m}}{(\lambda+2)\ldots(\lambda+2m)(\lambda+n+|\gamma|)\ldots(\lambda+n+|\gamma|+2m-2)}. \tag{15}$$

Formula (15) shows the structure of simple poles (9).

Lemma 1 Let $x \in R_+^n$, $n > 1$ and

$$E_\gamma^m(x) = \begin{cases} |x|^{2m-2}\, ln\,|x|, & 2m = n+|\gamma|; \\ \frac{|x|^{2m-n-|\gamma|}}{2m-n-|\gamma|}, & 2m < n+|\gamma|, \end{cases}$$

then for $|x| > \varepsilon$ $\forall\varepsilon > 0$ we get

$$\Delta_\gamma^m E_\gamma^m(x) = 0.$$

Proof. Let us first consider the case $n + |\gamma| > 2m$. We obtain

$$\Delta_\gamma E_\gamma^m(x) = \sum_{j=1}^n B_{\gamma_j}E_\gamma^m(x) = \sum_{j=1}^n \frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}E_\gamma^m(x) =$$

$$= \frac{1}{2m-n-|\gamma|}\sum_{j=1}^n \frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}|x|^{2m-n-|\gamma|} =$$

$$= \frac{1}{2m-n-|\gamma|}\sum_{j=1}^n \frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{2m-n-|\gamma|}{2}|x|^{2m-2-n-|\gamma|}2x_j =$$

$$= \sum_{j=1}^n \frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}|x|^{2m-2-n-|\gamma|}x_j^{1+\gamma_j} =$$

$$= \sum_{j=1}^n \frac{1}{x_j^{\gamma_j}}\left[\frac{2m-2-n-|\gamma|}{2}|x|^{2m-4-n-|\gamma|}2x_j^{2+\gamma_j} + (1+\gamma_j)|x|^{2m-2-n-|\gamma|}x_j^{\gamma_j}\right] =$$

$$= \sum_{j=1}^n[(2m-2-n-|\gamma|)|x|^{2m-4-n-|\gamma|}x_j^2 + (1+\gamma_j)|x|^{2m-2-n-|\gamma|}] =$$

$$= [(2m-2-n-|\gamma|)|x|^{2m-2-n-|\gamma|} + (n+|\gamma|)|x|^{2m-2-n-|\gamma|}] = (2m-2)|x|^{2m-2-n-|\gamma|}.$$

Continuing this process, we obtain the desired formula

$$\Delta_\gamma^m E_\gamma^m(x) = \frac{1}{2m-n-|\gamma|}\Delta_\gamma^m|x|^{2m-n-|\gamma|} = 0.$$

Now consider the case in which $n + |\gamma| = 2m$:

$$\Delta_\gamma E_\gamma^m(x) = \sum_{j=1}^n B_{\gamma_j}E_\gamma(x) = \sum_{j=1}^n \frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}|x|^{2m-2}\, ln\,|\, x| =$$

$$= \sum_{j=1}^n \frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\left[(2m-2)|x|^{2m-4}x_j + |x|^{2m-4}x_j\right] =$$

$$= (2m-1)\sum_{j=1}^n \frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}|x|^{2m-4}x_j^{1+\gamma_j} =$$

$$= (2m-1)\sum_{j=1}^{n}\frac{1}{x_j^{\gamma_j}}\left[\frac{2m-4}{2}|x|^{2m-6}2x_j^{2+\gamma_j}+(1+\gamma_j)|x|^{2m-4}x_j^{\gamma_j}\right]=$$

$$= (2m-1)\sum_{j=1}^{n}\left[(2m-4)|x|^{2m-6}x_j^2+(1+\gamma_j)|x|^{2m-4}\right]=$$

$$= (2m-1)[(2m-4)|x|^{2m-4}+(n+|\gamma|)|x|^{2m-4}] = 2(2m-1)(2m-2)|x|^{2m-4}.$$

Next

$$\Delta_\gamma^2 E_\gamma^m(x) = 2(2m-1)(2m-2)\Delta_\gamma |x|^{2m-4} =$$

$$= 2(2m-1)(2m-2)(2m-4)(2m-6+n+|\gamma|)|x|^{2m-6} =$$

$$= 2^2(2m-1)(2m-2)(2m-3)(2m-4)|x|^{2m-6}.$$

Continuing this process, we obtain $\Delta_\gamma^m |x|^{2m-2}\, ln\,|\,x| = 0$, when $n+|\gamma| = 2m$.

Theorem 1 Let $x \in R_+^n$, $n > 1$ and

$$E_\gamma^m(x) = \begin{cases} |x|^{2m-2}\, ln\,|x|, & m \geq \frac{n+|\gamma|}{2}\ and\ n+|\gamma|\ is\ even; \\ \frac{|x|^{2m-n-|\gamma|}}{2m-n-|\gamma|}, & in\ other\ cases, \end{cases} \tag{16}$$

is a weighted fundamental solution of the iterated Laplace-Bessel equation $\Delta_\gamma^m u = \delta_\gamma$ in D'_{ev}.

Proof. Applying Hankel transform (11) to $\Delta_\gamma^m u = \delta_\gamma$, using (13), we get $(-1)^m \rho^{2m} \boldsymbol{F}_\gamma u = 1$, $\rho^2 = \sum_{i=1}^{n} \xi_i^2$. Then $\boldsymbol{F}_\gamma u = (-1)^m \rho^{-2m}$ is a solution to $(-1)^m \rho^{2m} \boldsymbol{F}_\gamma u = 1$ for $2m < n+|\gamma|$, since $(-1)^m \rho^{-2m}$ is a locally integrable by R_+^n with the weight x^γ function.

Weighted generalised function ρ^λ has poles in

$$-(n+|\gamma|), -(n+|\gamma|)-2, -(n+|\gamma|)-4, \ldots,$$

when $n+|\gamma| \in N$. So if $m \geq \frac{n+|\gamma|}{2}$, $m \neq \frac{n+|\gamma|}{2}+p$, $p = 0,1,2,\ldots$ we can consider analytical continuation of ρ^{-2m} and by (13) get

$$u = \mathbf{F}_\gamma^{-1}[(-1)^m \rho^{-2m}](x) = \frac{(-1)^m 2^{n-|\gamma|}}{\prod_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \mathbf{F}_\gamma[\rho^{-2m}](x) =$$

$$= \frac{(-1)^m 2^{n-|\gamma|}}{\prod_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \mathrm{D}_{n,\gamma}(-m)|x|^{2m-n-|\gamma|} = \frac{(-1)^m 2^{n-2m}\Gamma\left(\frac{n+|\gamma|}{2}-m\right)}{(m-1)!\prod_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}|x|^{2m-n-|\gamma|}.$$

If $m \geq \frac{n+|\gamma|}{2}$ and $n+|\gamma|$ is even, so $2m$ is a pole of r^{2m}, then using Lemma 1 we get the second part of the formula (16).

Therefore, to each operator Δ_γ^m there corresponds a characteristic singular solution of the equation $\Delta_\gamma^m u = 0$ which is B -polyharmonic of least degree m in R_+^n and is given by (16).

Next, the solution to $\Delta_\gamma^m u = f$ in the form $(E_\gamma^m * f)_\gamma(x)$, where E_γ^m is the weighted fundamental solution (16) of Δ_γ^m.

Lemma 2 Let $\phi(x)$ and $\psi(x)$ be infinitely differentiable functions, even by each variable. Then for $k = 1,2,3,\ldots$

$$\Delta_\gamma^m(\phi\psi) = \sum_{k+i+j=m} 2^i \binom{k+j}{k}\binom{m}{k+j} \nabla^i \Delta_\gamma^k \phi \cdot \nabla^i \Delta_\gamma^j \psi \tag{17}$$

where

$$\nabla^i f\cdot\nabla^i g=\begin{cases} fg & if\ i=0;\\ \sum_{s=1}^n \frac{\partial f_s}{\partial x_s}\frac{\partial g_s}{\partial x_s} & if\ i<0.\end{cases}$$

Proof. We use an induction to prove the formula. Assume that $k=1$, it is clearly true since

$$\Delta_\gamma(\phi\psi)=\sum_{j=1}^n\left(\frac{\partial^2}{\partial x_j^2}+\frac{\gamma_j}{x_j}\frac{\partial}{\partial x_j}\right)(\phi\psi)=$$

$$=\sum_{j=1}^n\left(\phi\frac{\partial^2\psi}{\partial x_j^2}+2\frac{\partial\phi}{\partial x_j}\frac{\partial\psi}{\partial x_j}+\psi\frac{\partial^2\phi}{\partial x_j^2}+\phi\frac{\gamma_j}{x_j}\frac{\partial\psi}{\partial x_j}+\psi\frac{\gamma_j}{x_j}\frac{\partial\phi}{\partial x_j}\right)=$$

$$=\phi\Delta_\gamma\psi+2\nabla\phi\cdot\nabla\psi+\psi\delta_\gamma\phi=\sum_{m+i+j=1}2^i\binom{m+j}{m}\binom{1}{m+j}\nabla^i\Delta_\gamma^m\phi\cdot\nabla^i\Delta_\gamma^j\psi.$$

Indeed, when $m=1$, $i=j=0$ we get $\psi\Delta_\gamma\phi$, when $i=1, m=j=0$ we get $2\nabla\phi\cdot\nabla\psi$, when $j=1, m=i=0$ we get $\phi\Delta_\gamma\psi$.

Next let consider $\Delta_\gamma^{k+1}(\phi\psi)$:

$$\Delta_\gamma^{k+1}(\phi\psi)=\Delta_\gamma\Delta_\gamma^k(\phi\psi)=\Delta_\gamma\sum_{m+i+j=k}2^i\binom{m+j}{m}\binom{k}{m+j}\nabla^i\Delta_\gamma^m\phi\cdot\nabla^i\Delta_\gamma^j\psi=$$

$$=\sum_{m+i+j=k}2^i\binom{m+j}{m}\binom{k}{m+j}[\nabla^i\Delta_\gamma^{m+1}\phi\cdot\nabla^i\Delta_\gamma^j\psi+2\nabla^{i+1}\Delta_\gamma^m\phi\cdot\nabla^{i+1}\Delta_\gamma^j\psi+$$

$$+\nabla^i\Delta_\gamma^m\phi\cdot\nabla^i\Delta_\gamma^{j+1}\psi].$$

Replacing $(m+1)$ by m, we obtain

$$\sum_{m+i+j=k}2^i\binom{m+j}{m}\binom{k}{m+j}\nabla^i\Delta_\gamma^{m+1}\phi\cdot\nabla^i\Delta_\gamma^j\psi=$$

$$=\sum_{m+i+j=k+1}2^i\binom{m+j-1}{m-1}\binom{k}{m+j-1}\nabla^i\Delta_\gamma^m\phi\cdot\nabla^i\Delta_\gamma^j\psi.$$

Similarly,

$$\sum_{m+i+j=k}\quad 2^i\binom{m+j}{m}\binom{k}{m+j}\nabla^i\Delta_\gamma^m\phi\cdot\nabla^i\Delta_\gamma^{j+1}\psi=$$

$$=\sum_{m+i+j=k+1}\quad 2^i\binom{m+j-1}{m}\binom{k}{m+j-1}\nabla^i\Delta_\gamma^m\phi\cdot\nabla^i\Delta_\gamma^j\psi.$$

Finally,

$$\sum_{m+i+j=k}\quad 2^{i+1}\binom{m+j}{m}\binom{k}{m+j}\nabla^{i+1}\Delta_\gamma^m\phi\cdot\nabla^{i+1}\Delta_\gamma^j\psi=$$

$$=\sum_{m+i+j=k+1}\quad 2^i\binom{m+j}{m}\binom{k}{m+j}\nabla^i\Delta_\gamma^m\phi\cdot\nabla^i\Delta_\gamma^j\psi.$$

By direct calculation,

$$\binom{m+j-1}{m-1}\binom{k}{m+j-1}+\binom{m+j-1}{m}\binom{k}{m+j-1}+\binom{m+j}{m}\binom{k}{m+j}=$$

$$=\binom{m+j}{m}\binom{k+1}{m+j}.$$

So we get

$$\Delta_\gamma^{k+1}(\phi\psi) = \sum_{m+i+j=k+1} 2^i \binom{m+j}{m}\binom{k+1}{m+j} \nabla^i \Delta_\gamma^m \phi \cdot \nabla^i \Delta_\gamma^j \psi.$$

Lemma 2 was presented in [13] in the case then regular iterated Laplace operator acting to the product of two functions was considered, i.e. $\Delta^k(\phi\psi)$.

5. Taylor-Delsarte Formula

In 1938, Jean Delsarte (see [14, 15]) introduced a certain generalization of the notion of translation and, in connection with it, a corresponding generalization of Taylor's formula. The concept of generalized translation introduced by Delsarte was later examined from various perspectives by numerous authors (see Levitan [16, 17], Povzner [18], Bochner [19, 20], Weinberger [21], Hirschman [22]). Löfstöm and Peetre gave approximation theorems onnected with generalized translations in [23]. The article [24] presents the Taylor-Delsarte formula for the generalized Gegenbauer translation which is used to construct a modulus of smoothness, obtaining equivalent normalizations for associated functional spaces.

Taylor formula for regular shift is

$$f(x+a) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k f(x) a^k = e^{aD} f(x), \qquad D = \frac{d}{dx}. \tag{18}$$

In his papers [14, 15] Delsarte studies the generalized translation operator (see (6))

$$({}^\gamma T_x^y f)(x) = C(\gamma)\int_0^\pi f(\sqrt{x^2+y^2-2xy\cos\varphi})\sin^{\gamma-1}\varphi d\varphi, \quad C(\gamma) = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)}, \tag{19}$$

which solves the Cauchy problem

$$\begin{gathered} (B_\gamma)_x {}^\gamma T_x^y f(x) = (B_\gamma)_y {}^\gamma T_x^y f(x), \\ \gamma T_x^y f(x)|_{y=0} = f(x), \qquad \frac{\partial}{\partial y} {}^\gamma T_x^y f(x)|_{y=0} = 0, \end{gathered} \tag{20}$$

where B_γ is the Bessel operator $\gamma > 0$:

$$B_\gamma = (B_\gamma)_x = \frac{d^2}{dx^2} + \frac{\gamma}{x}\frac{d}{dx}.$$

Notice that for $\gamma = 0$ Bessel operator is the second derivative $(B_\gamma)_x = \frac{d^2}{dx^2} = D^2$ and

$$({}^0 T_x^y f)(x) = \frac{f(x+y)+f(x-y)}{2}.$$

Let $j_{\frac{\gamma-1}{2}}(\sqrt{\lambda}x)$ is a solution to the problem

$$\begin{gathered} B_\gamma u = -\lambda u, \qquad u = u(x) \\ u(0) = 1, \qquad u'(0) = 0. \end{gathered}$$

The function $j_{\frac{\gamma-1}{2}}(\sqrt{\lambda}x)$ has the form

$$j_{\frac{\gamma-1}{2}}(\sqrt{\lambda}x) = \frac{2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)}{(\sqrt{\lambda}x)^{\frac{\gamma-1}{2}}} J_{\frac{\gamma-1}{2}}(\sqrt{\lambda}x) = \sum_{k=0}^{\infty}(-1)^k \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{k!\,\Gamma\left(k+\frac{\gamma+1}{2}\right)}\left(\frac{x}{2}\right)^{2k}\lambda^k,$$

where J_α is the Bessel function of the first kind.

Taylor-Delsarte series for decomposition of translation (19) has the form

$$({}^\gamma T_x^y f)(x) = \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{k!\Gamma\left(k+\frac{\gamma+1}{2}\right)} \left(\frac{y}{2}\right)^{2k} (B_\gamma)^k f(x) = j_{\frac{\gamma-1}{2}}\left(i\sqrt{(B_\gamma)_x}\cdot y\right) f(x). \tag{21}$$

By analogy with (21) multidimensional Taylor-Delsarte series for decomposition of multidimensional translation (5) has the form

$$({}^\gamma \mathbf{T}_x^y f)(x) = \sum_{0\leq |2k|} \frac{1}{2^{2|k|}k!} \prod_{i=1}^{n} \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(k_i+\frac{\gamma_i+1}{2}\right)} y^{2k}\, (\mathrm{B}_\gamma)^k f(x), \tag{22}$$

where $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a multi-index consisting of positive fixed real numbers γ_i, $i = 1, \ldots, n$, $k = (k_1, \ldots, k_n)$ is a multi-index consisting of non-negative integers, $|k| = k_1 + \ldots + k_n$ is its length, $y^{2k} = y_1^{2k_1} \ldots y_n^{2k_n}$, $(B_\gamma)^k f(x) = (B_{\gamma_1})^{k_1} \ldots (B_{\gamma_n})^{k_n} f(x_1, \ldots, x_n)$, $B_{\gamma_i} = (B_{\gamma_i})_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$ and $k! = k_1! \cdot \ldots \cdot k_n!$.

6. Weighted Spherical Mean and Kipriyanov-Pizzetti's Formula

In his book [25], Fritz John considers the mean value operator on spheres in the Euclidean space R^n:

$$M(x,r,u) = \frac{1}{|S_n(1)|} \int_{S_n(1)} u(x+\beta r)dS, \tag{23}$$

where $S_n(1) = \{\, x \in R^n : |x| = 1 \,\}$ is the unit sphere centered at the origin, $r \geq 0$, $f(x+\beta r) = f(x_1+\beta_1 r, \ldots, x_n + \beta_n r)$, $|S_n(1)| = \int_{S_n} dS = 2\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$ is the area of the sphere $S_n(1)$, $\Gamma(\alpha)$ is the Euler gamma function, dS is an element of the surface $S_n(1)$.

The use of spherical means (23) of functions finds broad application in various branches of analysis, particularly in the theory of partial differential equations (see [12], p.74; [26], p.287).

Differential relations between spherical and solid means of continuous functions were derived in [27]. I. A. Kipriyanov in [11] extends Pizzetti's formula to weighted spherical mean at zero of functions within the corresponding weighted functional classes where instead of Laplace operator the mixed operator $\sum_{i=1}^{n} \quad \frac{\partial^2}{\partial x_i^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}$ was used. The formula presented in [11] (p. 118, formula (2.13)) is known as the Kipriyanov-Pizzetti formula. Further generalizations of the Kipriyanov-Pizzetti formula are related to the study of problems involving the Laplace-Bessel operator. We study Pizzetti's formula adapted to the case where the Besel operator acts on all variables.

Weighted spherical mean of function $f(x)$, $x \in \overline{R}_+^n$ for $n \geq 2$ is

$$(M_t^\gamma f)(x) = \frac{1}{|S_1^+(n)|_\gamma} \int_{S_1^+(n)} ({}^\gamma \mathbf{T}_x^{t\theta} f)(x) \theta^\gamma dS, \tag{24}$$

where $\theta^\gamma = \prod_{i=1}^{n} \quad \theta_i^{\gamma_i}$, $S_1^+(n) = \{\theta : |\theta| = 1, \theta \in R_+^n\}$ is a part of a sphere in R_+^n, and $|S_1^+(n)|_\gamma$ is given by (4). For $n = 1$ let $M_t^\gamma[f(x)] = ({}^\gamma T_x^t f)(x)$.

In polar coordinates, where $x = r\theta$ with $r = |x|$ and $\theta \in S_1^+(n)$, the functional (8) can be expressed as follows:

$$(r^\lambda, \phi)_\gamma = \int_0^\infty \quad r^{\lambda+n+|\gamma|-1} \left(\int_{S_1^+(n)} \quad \phi(r\theta)\theta^\gamma dS \right) dr = |S_1^+(n)|_\gamma \int_0^\infty \quad r^{\lambda+n+|\gamma|-1} (M_r^\gamma \phi)(0) dr,$$

where

$$(M_r^\gamma \phi)(0) = \frac{1}{|S_1^+(n)|_\gamma} \int_{S_1^+(n)} \phi(r\theta)\theta^\gamma dS. \tag{25}$$

It is important to note that when $\phi \in D_{ev}$ the function M_r^γ is infinitely differentiable by r for $r \geq 0$ and decreases more rapidly than any power of $\frac{1}{r}$ as $r \to \infty$. This behavior is due to similar properties of the function $\phi(x)$.

Theorem 2 Let $u \in C_{ev}^\infty(\overline{\Omega}_+)$, $x \in \overline{\Omega}_+$, then for t the next formula is valid

$$(M_t^\gamma u)(x) = \sum_{p=0}^\infty \frac{\Delta_\gamma^p u(x)}{2^{2p} p! \left(\frac{n+|\gamma|}{2}\right)_p} t^{2p}, \tag{26}$$

where $(a)_p = a(a+1)\dots(a+p-1)$ is the Pochhammer symbol.

Proof. By definition, using multidimensional Taylor-Delsarte series (22), we will have

$$(M_t^\gamma u)(x)] = \frac{1}{|S_1^+(n)|_\gamma} \int_{S_1^+(n)} ({}^\gamma \mathbf{T}_x^{t\theta} u)(x)\theta^\gamma dS =$$

$$= \frac{1}{|S_1^+(n)|_\gamma} \sum_{0\leq|k|} \frac{t^{2|k|}}{2^{2|k|}k!} \prod_{i=1}^n \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(k_i+\frac{\gamma_i+1}{2}\right)} (\mathrm{B}_\gamma)^k u(x) \int_{S_1^+(n)} \theta^{\gamma+2k} dS.$$

Applying formula (4), we obtain

$$(M_t^\gamma u)(x) = \frac{1}{|S_1^+(n)|_\gamma} \sum_{0\leq|k|} \frac{t^{2|k|}}{2^{2|k|}k!} \prod_{i=1}^n \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(k_i+\frac{\gamma_i+1}{2}\right)} (\mathrm{B}_\gamma)^k u(x) \frac{\prod_{i=1}^n \Gamma\left(k_i+\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}+|k|\right)} =$$

$$= \Gamma\left(\frac{n+|\gamma|}{2}\right) \sum_{0\leq|k|} \frac{(\mathrm{B}_\gamma)^k u(x)}{2^{2|k|}k!\Gamma\left(\frac{n+|\gamma|}{2}+|k|\right)} t^{2|k|} = \sum_{0\leq|k|} \frac{(\mathrm{B}_\gamma)^k u(x)}{2^{2|k|}k!\left(\frac{n+|\gamma|}{2}\right)_{|k|}} t^{2|k|}.$$

Dividing the sum by two and applying formula (14), we get

$$(M_r^\gamma u)(x) = \sum_{p=0}^\infty \sum_{|k|=p} \frac{(\mathrm{B}_\gamma)^k u(x)}{2^{2|k|}k!\left(\frac{n+|\gamma|}{2}\right)_{|k|}} r^{2|k|} = \sum_{p=0}^\infty \frac{r^{2p}}{2^{2p}p!\left(\frac{n+|\gamma|}{2}\right)_p} \sum_{|k|=p} \frac{p!}{k!} (\mathrm{B}_\gamma)^k u(x) =$$

$$= \sum_{p=0}^\infty \frac{\Delta_\gamma^p u(x)}{2^{2p}p!\left(\frac{n+|\gamma|}{2}\right)_p} r^{2p}.$$

Formula (26) can be written in the form (compare with the formula (21))

$$(M_r^\gamma u)(x)] = \sum_{p=0}^\infty \frac{\Gamma\left(\frac{n+|\gamma|}{2}\right)}{p!\Gamma\left(p+\frac{n+|\gamma|}{2}\right)} \left(\frac{r}{2}\right)^{2p} (\Delta_\gamma)^p u(x) = j_{\frac{n+|\gamma|}{2}-1}\left(i\sqrt{(\Delta_\gamma)_x} \cdot r\right) u(x). \tag{27}$$

Here and further the right-hand side is, of course, to be interpreted operationally.

The next corollary is valid.

Corollary 1 Let $u \in C_{ev}^\infty(\overline{\Omega}_+)$, $x \in \overline{\Omega}_+$, then for $r > 0$ the next formula is valid

$$(M_r^\gamma u)(0) = |S_1^+(n)|_\gamma \sum_{p=0}^\infty \frac{\Delta_\gamma^p u(0)}{2^{2p}p!\left(\frac{n+|\gamma|}{2}\right)_p} r^{2p}, \tag{28}$$

where $(a)_p = a(a+1)\dots(a+p-1)$ is the Pochhammer symbol.

B -harmonic functions i.e. solutions of the Laplace-Bessel equation $\Delta_\gamma u = 0$ possess the mean value property with weighted spherical mean.

Proposition 1 The value of the B -harmonic on $B_r^+(n)$ function $u=u(x)$ at the point $x\in B_r^+(n)$ is equal to the weighted spherical mean value of the function f on the boundary $B_r^+(n)$:

$$(M_r^\gamma u)(x)=u(x).$$

Proof. It function $u(x)$ is B -harmonic on $B_r^+(n)$, then it satisfies the equation $\Delta_\gamma u=0$. in $B_r^+(n)$ Therefore, from the representation (27) of the weighted spherical mean operator it follows that M_t^γ of the B -harmonic function is equal to $(M_r^\gamma u)(x)=u(x)$.

7. Applications to PDEs with Laplace-Bessel Operator

P. Pizzetti derived a formula in [1, 2] for expanding spherical means as a series in terms of the Laplace operator, applicable to sufficiently smooth functions. Pizzetti's formula (1) explicitly demonstrates the equivalence between rotation-invariant integration over a sphere and the application of rotation-invariant differential operators. Whenever the Laplace–Bessel operator appears in the equation, by transitioning to spherical coordinates in its solution, we obtain a weighted spherical mean and can apply formula (26).

By analogy with the classical cases, we consider B -parabolic, B -hyperbolic, and B -elliptic partial differential equations involving the Laplace–Bessel operator.

Proposition 2 Let $f=f(x)\in C_{ev}^\infty$, $x\in R_+^n$, then the unique solution to the Cauchy problem for B -parabolic equation

$$\begin{gathered}u_t=(\Delta_\gamma)_x u,\quad u=u(x,t),\quad x\in \mathrm{R}_+^n,\quad t>0;\\ u(x,0)=f(x)\end{gathered}\tag{29}$$

is

$$u(x,t)=\sum_{p=0}^\infty \frac{t^p}{p!}\Delta_\gamma^p f(x)=e^{t(\Delta_\gamma)_x}f(x).$$

Proof. In [28] was shown that

$$u(x,t)=\frac{t^{-\frac{n+|\gamma|}{2}}}{2^{|\gamma|}\prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}\int_{R_+^n} e^{-\frac{|y|^2}{4t}}({}^\gamma \boldsymbol{T}_x^y f)(x)y^\gamma dy.$$

Using the spherical coordinates $y=r\theta$, we obtain

$$u(x,t)=\frac{t^{-\frac{n+|\gamma|}{2}}}{2^{|\gamma|}\prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}\int_0^\infty e^{-\frac{r^2}{4t}}r^{n+|\gamma|-1}dr\int_{S_1^+(n)}({}^\gamma \mathbf{T}_x^{r\theta} f)(x)\theta^\gamma dS=$$

$$=\frac{t^{-\frac{n+|\gamma|}{2}}}{2^{|\gamma|}\prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}|S_1^+(n)|_\gamma\int_0^\infty e^{-\frac{r^2}{4t}}r^{n+|\gamma|-1}\,(M_r^\gamma f)(x)dr.$$

Using formula (26) we get

$$u(x,t)=\frac{t^{-\frac{n+|\gamma|}{2}}}{2^{|\gamma|}\prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}|S_1^+(n)|_\gamma\sum_{p=0}^\infty \frac{\Delta_\gamma^p f(x)}{2^{2p}p!\left(\frac{n+|\gamma|}{2}\right)_p}\int_0^\infty e^{-\frac{r^2}{4t}}r^{n+|\gamma|+2p-1}dr=$$

$$=\frac{2^{n-1}}{\prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}|S_1^+(n)|_\gamma\sum_{p=0}^\infty \frac{\Delta_\gamma^p f(x)}{p!\left(\frac{n+|\gamma|}{2}\right)_p}\Gamma\left(p+\frac{n+|\gamma|}{2}\right)t^p=$$

$$=\frac{2^{n-1}}{\prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}\frac{\prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\sum_{p=0}^\infty \frac{\Gamma\left(\frac{n+|\gamma|}{2}\right)}{p!\Gamma\left(p+\frac{n+|\gamma|}{2}\right)}\Gamma\left(p+\frac{n+|\gamma|}{2}\right)t^p\Delta_\gamma^p f(x)=$$

$$= \sum_{p=0}^{\infty} \frac{t^p}{p!} \Delta_\gamma^p f(x) = e^{t(\Delta_\gamma)_x} f(x).$$

Proposition 3 For $k \in R$, $k > n + |\gamma| - 1$, $f = f(x) \in C_{ev}^{\infty}$, $x \in R_+^n$ the unique solution to the first Cauchy problem for general Euler-Darboux-Poisson equation

$$\begin{cases} (B_k)_t = (\Delta_\gamma)_x u, \quad u = u(x,t;k), \quad x \in \mathrm{R}_+^n, \quad t > 0, \quad k \in \mathrm{R}; \\ u(x,0;k) = f(x), \quad u_t(x,0;k) = 0 \end{cases} \tag{30}$$

is

$$u(x,t;k) = \sum_{p=0}^{\infty} \frac{\Gamma\left(\frac{k+1}{2}\right)}{p!\Gamma\left(p+\frac{k+1}{2}\right)} \left(\frac{t}{2}\right)^{2p} \Delta_\gamma^p f(x) = j_{\frac{k-1}{2}}\left(i\sqrt{(\Delta_\gamma)_x} \cdot t\right) f(x).$$

Proof. In [29] was shown that for $k \in R$, $k > n + |\gamma| - 1$, $f = f(x) \in C_{ev}^2$, $x \in R_+^n$ the unique solution to the first Cauchy problem (30) is

$$u(x,t;k) = C_{n,\gamma,k} \mathrm{M}_t^{\alpha,\gamma} f(x),$$

where

$$\mathrm{M}_t^{\alpha,\gamma} f(x) = |S_1^+(n)|_\gamma \int_0^1 (1-\lambda^2)^{\alpha-1} \lambda^{n+|\gamma|-1} (M_{t\lambda}^\gamma f)(x) d\lambda, \tag{31}$$

$$\alpha = \frac{k-n-|\gamma|+1}{2}, \; C_{n,\gamma,k} = \frac{2^n \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right) \prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}.$$

The unique solution of the problem (30) for $k = n + |\gamma| - 1$ is the weighted spherical mean $M_t^\gamma f(x)$.

Using formula (26) we get (compare with (27))

$$\mathrm{M}_t^{\alpha,\gamma} f(x) = |S_1^+(n)|_\gamma \int_0^1 (1-\lambda^2)^{\alpha-1} \lambda^{n+|\gamma|-1} (M_{t\lambda}^\gamma f)(x) d\lambda =$$

$$= |S_1^+(n)|_\gamma \sum_{p=0}^{\infty} \frac{\Delta_\gamma^p f(x)}{2^{2p} p! \left(\frac{n+|\gamma|}{2}\right)_p} t^{2p} \int_0^1 (1-\lambda^2)^{\alpha-1} \lambda^{n+|\gamma|+2p-1} d\lambda =$$

$$= \frac{\Gamma(\alpha)\Gamma\left(\frac{n+|\gamma|}{2}\right)}{2\Gamma\left(\frac{k+1}{2}\right)} |S_1^+(n)|_\gamma \sum_{p=0}^{\infty} \frac{\Gamma\left(\frac{k+1}{2}\right)}{p!\Gamma\left(p+\frac{k+1}{2}\right)} \left(\frac{t}{2}\right)^{2p} \Delta_\gamma^p f(x) =$$

$$= \frac{\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}\right)}{2\Gamma\left(\frac{k+1}{2}\right)} |S_1^+(n)|_\gamma j_{\frac{k-1}{2}}\left(i\sqrt{(\Delta_\gamma)_x} \cdot t\right) f(x)$$

and

$$u(x,t;k) = \frac{2^n \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right) \prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)} \times$$

$$\times \frac{\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}\right)}{2\Gamma\left(\frac{k+1}{2}\right)} \frac{\prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)} j_{\frac{k-1}{2}}\left(i\sqrt{(\Delta_\gamma)_x} \cdot t\right) f(x) =$$

$$= j_{\frac{k-1}{2}}\left(i\sqrt{(\Delta_\gamma)_x} \cdot t\right) f(x).$$

Proposition 4 Let $x \in R_n^+$, $n > 1$, $k > n + |\gamma| - 3$, $f = f(x) \in C_{ev}^{\infty}$ then the unique solution to the problem

$$\begin{cases} (B_k)_t u + \Delta_\gamma u = b^2 u, \quad u = u(x,t;k), \qquad b > 0, \quad x \in \mathrm{R}_+^n, \quad t > 0, \quad k \in \mathrm{R}; \\ u(x,0;k) = f(x) \end{cases} \tag{32}$$

has the form

$$u(x,t;k)=\frac{2^{\frac{k}{2}}b^{1-\frac{k}{2}}}{\Gamma\left(\frac{1-k}{2}\right)}\sum_{p=0}^{\infty}\frac{\Gamma\left(p+\frac{n+|\gamma|-1}{2}\right)}{p!\Gamma\left(p+\frac{n+|\gamma|}{2}\right)}K_{\frac{k}{2}+p-1}(bt)\left(\frac{t}{2b}\right)^{p}\Delta_{\gamma}^{p}f(x).$$

Proof. In [30] was shown that

$$u(x,t;k)=\frac{2^{\frac{n+1-|\gamma|+k}{2}}b^{\frac{n+1+|\gamma|-k}{2}}t^{\frac{1-k}{2}}}{\Gamma\left(\frac{1-k}{2}\right)\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\times$$

$$\times\int_{\mathrm{R}_+^n}({}^{\gamma}\mathbf{T}_x^y f)(x)\left(\sqrt{|y|^2+t^2}\right)^{\frac{k-n-1-|\gamma|}{2}}K_{\frac{n+1+|\gamma|-k}{2}}\left(b\sqrt{|y|^2+t^2}\right)y^{\gamma}dy$$

is the unique solution to the problem (32). Using the spherical coordinates $y=r\theta$, we obtain

$$u(x,t;k)=\frac{2^{\frac{n+1-|\gamma|+k}{2}}b^{\frac{n+1+|\gamma|-k}{2}}t^{\frac{1-k}{2}}}{\Gamma\left(\frac{1-k}{2}\right)\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\times$$

$$\times\int_0^{\infty}\left(\sqrt{r^2+t^2}\right)^{\frac{k-n-1-|\gamma|}{2}}K_{\frac{n+1+|\gamma|-k}{2}}\left(b\sqrt{r^2+t^2}\right)r^{n+|\gamma|-1}dr\int_{S_1^+(n)}({}^{\gamma}\mathbf{T}_x^{r\theta}f)(x)\theta^{\gamma}dS=$$

$$=\frac{2^{\frac{n+1-|\gamma|+k}{2}}b^{\frac{n+1+|\gamma|-k}{2}}t^{\frac{1-k}{2}}}{\Gamma\left(\frac{1-k}{2}\right)\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}|S_1^+(n)|_{\gamma}\times$$

$$\times\int_0^{\infty}\left(\sqrt{r^2+t^2}\right)^{\frac{k-n-1-|\gamma|}{2}}K_{\frac{n+1+|\gamma|-k}{2}}\left(b\sqrt{r^2+t^2}\right)r^{n+|\gamma|-1}(M_r^{\gamma}f)(x)dr.$$

Using formula (26) we get

$$u(x,t;k)=\frac{2^{\frac{n+1-|\gamma|+k}{2}}b^{\frac{n+1+|\gamma|-k}{2}}t^{\frac{1-k}{2}}}{\Gamma\left(\frac{1-k}{2}\right)\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}|S_1^+(n)|_{\gamma}\sum_{p=0}^{\infty}\frac{\Delta_{\gamma}^{p}f(x)}{2^{2p}p!\left(\frac{n+|\gamma|}{2}\right)_p}\times$$

$$\times\int_0^{\infty}\left(\sqrt{r^2+t^2}\right)^{\frac{k-n-1-|\gamma|}{2}}K_{\frac{n+1+|\gamma|-k}{2}}\left(b\sqrt{r^2+t^2}\right)r^{n+|\gamma|+2p-1}dr=$$

$$=\frac{2^{\frac{k}{2}}b^{1-\frac{k}{2}}}{\Gamma\left(\frac{1-k}{2}\right)}\sum_{p=0}^{\infty}\frac{\Gamma\left(p+\frac{n+|\gamma|-1}{2}\right)}{p!\Gamma\left(p+\frac{n+|\gamma|}{2}\right)}K_{\frac{k}{2}+p-1}(bt)\left(\frac{t}{2b}\right)^{p}\Delta_{\gamma}^{p}f(x).$$

Conflict of Interest

The author declares that there are no conflicts of interest or potential biases related to the research presented in this study.

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