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Volterra Control Systems with Risk-Sensitive Performance

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ABSTRACT

In the present paper, our aim is to focus on the problem of optimal control under a risk-sensitive performance functional, where the system is given by a forward stochastic differential volterra equation. The risk-neutral control system has been used as a preliminary step, where the set of admissible controls is convex, and the optimal solution exists, the necessary as well as sufficient optimality conditions for risk-sensitive performance are proved.

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1. Introduction

Stochastic Differential Volterra Equations (SDVE in short) are a class of stochastic differential equations (SDE) where the drift and diffusion coefficients depend on the past history of the process, rather than just its current state. This makes them a natural extension of standard SDE, incorporating memory effects commonly seen in applications such as finance, biology, and physics.

A stochastic maximum principle (in short SMP) for a class of risk-sensitive mean-field control problems has been proved by Djehiche *et al.* in *DTT*. Building on their contribution, we address the present problem. In their work, the drift, the diffusion and the terminal cost functions depend on the state, the control. This work extends the results of Lim *et al.* in *Lim – Zhou* to risk-sensitive control problems for dynamics that are non-Markovian and of mean-field type. Also based on our previous research of Chala *et al.* in [6; 9; 10] to derive the risk sensitive SMP of optimal control problems for non-Markovian dynamics which may be time-inconsistent in the sense that the Bellman optimality principle does not hold.

In the risk-sensitive performance case, the system is governed by a SDVE

$$\begin{cases} dx_t^v &= b(t, t, x_t^v, v_t)dt + \sigma(t, t, x_t^v, v_t)dW_t \\ &+ \left(\int_0^t \frac{\partial b}{\partial t}(t, s, x_s^v, v_s)ds \right)dt + \left(\int_0^t \frac{\partial \sigma}{\partial t}(t, s, x_s^v, v_s)dW_s \right)dt, \\ x_0^v &= \alpha, \end{cases}$$

where b and σ are given functions, the initial condition α is F_0 –measurable and square integrable, and $W = (W_t)_{t \geq 0}$ is a standard Brownian motion defined on a filtered probability space $(\Omega, F, (F_t)_{t \geq 0}, P)$, satisfying the usual conditions.

The control variable $v = (v_t)$, called strict control, is an F_t –adapted process with values in some set U of R . We denote by U the class of all strict admissible controls. The criteria to be maximized over U has the form

$$J^\theta(v) = \mathbb{E} \left[\exp \theta \left(\int_0^T l(t, x_t^v, v_t)dt + \Psi(x_T^v) \right) \right],$$

where l and Ψ are given maps and x_t^v is the trajectory controlled by v . We denote by E the expectation with respect to P .

A control $u \in U$ is called optimal if it satisfies

$$J^\theta(u) = \sup_{v \in U} J^\theta(v).$$

To achieve the objective of this paper and establish the necessary as well as sufficient optimality conditions, the existence and uniqueness of the optimal control which maximize the functional cost is proved by [3; 15; 16; 18]

We transform the intermediate adjoint processes to a simpler form using the fact that the set of admissible controls is convex. We proceed with our research in this paper as follows. Firstly, we give the optimality conditions for risk-neutral controls. The idea is to use the fact that the auxiliary state process ξ_t^v is the best intermediate step to translate our initial control problem $\{(6), (7), (8)\}$ into the system of forward SDVE see (9) in section 3. Secondly, we suggest a transformation of the adjoint processes (p_1, q_1) and (p_2, q_2) into the following adjoint equations $(\tilde{p}_1(t), 0)$ and $(\tilde{p}_2(t), \tilde{q}_2(t, t))$, where the process $\tilde{p}_1(t)$ is still a square integrable martingale, which would mean that $\tilde{p}_1(t) = \tilde{p}_1(0)$, and is equal to the constant $E[\tilde{p}_1(0)]$. Then by applying the result obtained by both Yong *et al.* [18] and El-Karoui *et al.*, [8] but with some additional ideas, we use this transformation and in virtue of the logarithmic transformation given in the Lemma 3.4 to solve this problem and drive the necessary as well as sufficient optimality conditions of the type risk-sensitive performance cost.

Our goal in this paper is to establish a set of necessary as well as sufficient optimality conditions for SDVE for risk-sensitive performance cost. The backward problem under consideration is not a simple extension from the

mathematical point of view, but also provides interesting models in many applications such as mathematical economics or optimal control. The proof of our main result is based on the spike variation method based on Theorem 3.1 as a preliminary step for the risk-neutral control problem.

The paper is organized as follows: In section 2, we give the precise problem formulations, and introduce the risk-sensitive model and present the various assumptions used throughout this paper. In section 3, we shall study our system of two forward SDVEs, the new approach method transformation of the adjoint process is given and studied, SMP for risk-neutral is given, which will be the main result in the next part, we give our first main result, the necessary optimality conditions for risk-sensitive control problem under an additional hypothesis is established, and the sufficient optimality conditions for risk-sensitive performance cost as our second main result is obtained under the convexity of the Hamiltonian function. The conclusion and future directions are given in the last section (section 4).

2. Problem Formulation

2.1. Spaces

In all what follows, we will work on the classical probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$, such that $\mathcal{F}_0$ contains all the P –null sets, $\mathcal{F}_T = \mathcal{F}$ for an arbitrarily fixed time horizon T , and $(\mathcal{F}_t)_{t \leq T}$ satisfies the usual conditions. Let T be a strictly positive real number.

Notation 2.1 We need to define some additional notations. Given $s \leq t$, let us introduce the following spaces:

$S^2([0, T], \mathbb{R})$ is the set of $\mathbb{R}$ –valued $\mathcal{F}$ –adapted cadl g processes $(p(t))_{t \in [0, T]}$ such that

$$\|p\|_{S^2([0, T], \mathbb{R})} := E \left[\int_0^T |p(t)|^2 dt \right]^{\frac{1}{2}} < \infty.$$

$M^2([0, T]^2, \mathbb{R})$ is the set of $\mathbb{R}$ –valued $\mathcal{F}$ –adapted processes $(Q(t, s))_{(t, s) \in [0, T]^2}$ such that

$$\|Q\|_{M^2([0, T]^2, \mathbb{R})} := E \left[\int_0^T \int_0^T |Q(t, s)|^2 ds dt \right]^{\frac{1}{2}} < \infty.$$

we denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$.

2.2. Generalized Hida-Malliavin Derivative

The Malliavin derivative D_t was originally introduced by Malliavin [13] as a stochastic calculus of variation used to prove results about smoothness of densities of solutions of stochastic differential equations in $\mathbb{R}$ driven by Brownian motion. The domain of definition of the Malliavin derivative is a subspace $D_{1,2}$ of $M^2(P)$. We refer to Di Nunno *et al.* [14] for information about the Malliavin derivative D_t for Brownian motion and, more generally, L vy processes. Subsequently, in Aase *et al.* [1] the Malliavin derivative was put into the context of the white noise theory of Hida and extended to an operator defined on the whole of $M^2(P)$ and with values in the Hida space $(\mathcal{S})^*$ of stochastic distributions. This extension is called the Hida-Malliavin derivative.

There are several advantages with working with this extended Hida-Malliavin derivative:

- The Hida-Malliavin derivative is defined on all of $M^2(P)$, and it is an extension of the classical Malliavin derivative, in the sense that it coincides with the classical Malliavin derivative on the subspace $D_{1,2}$.
- The Hida-Malliavin derivative combines well with the white noise calculus, including the Skorohod integral and calculus with the wick product.

Moreover, in the following, we let $(\mathcal{S})^*$ denote the Hida space of stochastic distributions.

It was proved in Aase *et al.* [1] that one can extend the Hida-Malliavin derivative operator D_t from $D_{1,2}$ to all of

$M^2(F_T, P)$ in such a way that, also denoting the extended operator by D_t , for all $F \in M^2(F_T, P)$, we have

$$D_t F \in (S)^* \text{ and } (t, \omega) \rightarrow \mathbb{E}[D_t F | F_t] \text{ belongs to } M^2(\lambda \times P),$$

where λ is Lebesgue measure on $[0, T]$.

Proposition 2.1 (Generalized Clark-Ocone formula)

For all $F \in M^2(F_T, P)$, we have

$$F = E[F] + \int_0^T E[D_t F | F_t] dW_t. \quad (1)$$

Moreover, we have the following generalized duality formula, for the Brownian motion:

Proposition 2.2 (The generalized duality formula for W)

Fix $s \in [0, T]$. If $t \rightarrow \varphi(t, s, \omega) \in M^2(\lambda \times P)$ is F -adapted with $E \left[\int_0^T \varphi^2(t, s) dt \right] < \infty$ and $F \in M^2(F_T, P)$, then we have,

$$\mathbb{E} \left[F \int_0^T \varphi(t, s) dW_t \right] = \mathbb{E} \left[\int_0^T \mathbb{E}[D_t F | F_t] \varphi(t, s) dt \right]. \quad (2)$$

Proof. As it has observed by Agram and Øksendal *agram*, for fixed $s \in [0, T]$, by the equalities (1 – 2) and the Itô isometry, we get

$$E \left[F \int_0^T \varphi(t, s) dW_t \right] = E \left[\left(E[F] + \int_0^T E[D_t F | F_t] dW_t \right) \left(\int_0^T \varphi(t, s) dW_t \right) \right] = E \left[\int_0^T E[D_t F | F_t] \varphi(t, s) dt \right].$$

The following result is the Hida-Malliavin representation for BSVIE:

Theorem 2.1 (Representation theorem for BSVIE) Suppose that the driver $f(t, s, p, q) \in [0, T]^2 \times R \times R \rightarrow R$ is F -adapted with respect to s for all t, p, q and that $s \rightarrow (p(s), q(t, s)) \in M^2 \times M^2$ are given F -adapted processes with respect to $s \in [t, T]$, and they satisfy

$$p(t) = F(t) - \int_t^T f(t, s, p(s), q(t, s)) ds + \int_t^T q(t, s) dW_s, t \in [0, T]. \quad (3)$$

where $F(t) \in M^2(F_T, P, R)$. Then for *a. a.* t and s , the following holds:

$$q(t, s) = E[D_s p(t) | F_s], s < t. \quad (4)$$

Proof. We know by Theorem (3.1) in Agram *et al.* *AGRAM12* that for a Lipschitz driver f and for a terminal value $F(t) \in M^2(F_T, P, R)$, the above BSVIE 3 has a unique solution. Moreover, for all $t \in [0, T]$, it holds that

$$p(t) = E[p(t)] + \int_0^t q(t, s) dW_s, t \in [0, T]. \quad (5)$$

For more details, we refer to Yong *yon* (for the Brownian framework). Taking the Hida-Malliavin derivatives of (5), we get, for $s < t$,

$$E[D_s p(t) | F_s] = q(t, s) + E \left[\int_s^t D_s p(t) dW_s | F_s \right] = q(t, s).$$

2.3. Assumptions for the System Forward SDVE

Definition 2.1 An admissible control is a U -valued measurable F_t -adapted process v , such that $\|v\|_{S^2([0, T], R)} < \infty$.

We denote by U the set of all admissible controls, and we assume the set U of all admissible controls is convex.

For all $v \in U$, we consider the following stochastic differential Volterra equation system

$$x_t^v = \alpha + \int_0^t b(t, s, x_s^v, v_s) ds + \int_0^t \sigma(t, s, x_s^v, v_s) dW_s, \quad (6)$$

By differentiability form, we get

$$\begin{cases} dx_t^v &= b(t, t, x_t^v, v_t) dt + \sigma(t, t, x_t^v, v_t) dW_t \\ &+ \left(\int_0^t \frac{\partial b}{\partial t}(t, s, x_s^v, v_s) ds \right) dt + \left(\int_0^t \frac{\partial \sigma}{\partial t}(t, s, x_s^v, v_s) dW_s \right) dt, \\ x_0^v &= \alpha, \end{cases}$$

where $b: [0, T]^2 \times R \times U \rightarrow R$ and $\sigma: [0, T]^2 \times R \times U \rightarrow R$, α is F_0 –measurable and square integrable.

We define now the following risk-sensitive performance functional type with terminal cost

$$J^\theta(v) = E \left[\exp \theta \left\{ \Psi(x_T^v) + \int_0^T l(t, x_t^v, v_t) dt \right\} \right], \quad (7)$$

where $\Psi: R \rightarrow R$ and $l: [0, T] \times R \times U \rightarrow R$, and $\theta > 0$ is called the risk-sensitive parameter.

Strict control $u \in U$ is an optimal control if it satisfies

$$J^\theta(u) = \sup_{v \in U} J^\theta(v) \quad (8)$$

A control that solves the problem $\{(6), (7), (8)\}$ is called optimal. Our goal is to establish risk-sensitive necessary as well as sufficient conditions of optimality, satisfied by a given optimal control, in the form of risk-sensitive stochastic maximum principle with a risk-sensitive performance cost type.

Suppose the following assumptions

Assumption 2.1.

1. The functions $b(t, s, x, v)$ and $\sigma(t, s, x, v)$ are F_s –predictable for $s \leq t$, for each (t, x, v) .
2. There exists a constant $C > 0$, such that, for all $0 \leq s \leq t \leq T, v \in U$ and all b, σ satisfy

$$|b(t, s, 0, v)| + |\sigma(t, s, 0, v)| \leq C.$$

3. The functions $b(t, s, x, v)$ and $\sigma(t, s, x, v)$ are Lipschitz continuous with respect to x uniformly in t, s, u , i.e. for all $x, x' \in R$, we have

$$|b(t, s, x, v) - b(t, s, x', v)| + |\sigma(t, s, x, v) - \sigma(t, s, x', v)| \leq C|x - x'|.$$

4. The functions $b(t, s, x, v)$ and $\sigma(t, s, x, v)$ are linear growth with respect to x uniformly in t, s and v , i.e. for all $x, x' \in R$, we have

$$|b(t, s, x, v)| + |\sigma(t, s, x, v)| \leq C|1 + |x||.$$

Assumption 2.2.

1. The processes b, σ and l are F_s –adapted for all $s \leq t$, and twice continuously differentiable C^2 with respect to t, x and continuously differentiable C^1 with respect to u for each s .
2. The terminal cost function Ψ is assumed to be F_T –measurable and C^1 in x .

3. All the partial derivatives are supposed to be bounded by $\mathcal{C}(1 + |x|)$.

Theorem 2.2 Under the above assumptions (1 – 2), for every $v \in U$ the system (6) admits a unique solution, and the cost function (7) is well defined from U into R .

3. Auxiliary Process and Pontryagin's Maximum Principle for Risk-Sensitive Control

As we know, there is no solution of such problem of control with exponential utility, for this reason we may introduce the auxiliary process ξ^v which is the solution of SDE

$$d\xi_t^v = l(t, x_t^v, v_t)dt, \quad \xi_0^v = 0.$$

Then our control problem $\{(6), (7), (8)\}$ is equivalent to

$$\begin{cases} \sup_{v \in U} \mathbb{E}[\exp\theta\{\Psi(x_T^v) + \xi_T^v\}] = \sup_{v \in U} \mathbb{E}[\varphi(x_T^v, \xi_T^v)], \\ \text{subject to} \\ \xi_t^v = \int_0^t l(s, x_s^v, v_s)ds, \\ x_t^v = \alpha + \int_0^t \sigma(s, x_s^v, v_s)dW_s, \\ x_0^v = \alpha, \xi_0^v = 0. \end{cases} \quad (9)$$

By differentiability form, we get

$$\begin{cases} \sup_{v \in U} \mathbb{E}[\exp\theta\{\Psi(x_T^v) + \xi_T^v\}] = \sup_{v \in U} \mathbb{E}[\varphi(x_T^v, \xi_T^v)], \\ \text{subject to} \\ d\xi_t^v = l(t, x_t^v, v_t)dt, \\ dx_t^v = b(t, x_t^v, v_t)dt + \sigma(t, x_t^v, v_t)dW_t + \left(\int_0^t \frac{\partial b}{\partial t}(t, s, x_s^v, v_s)ds \right)dt + \left(\int_0^t \frac{\partial \sigma}{\partial t}(t, s, x_s^v, v_s)dW_s \right)dt, \\ x_0^v = \alpha, \xi_0^v = 0. \end{cases}$$

We define $A_T^\theta := \exp \theta \theta_T$, where

$$\theta_T := \Psi(x_T^v) + \int_0^T l(t, x_t^v, v_t)dt.$$

Lemma 3.1 We suppose that the assumption (2.2) holds, then the risk-sensitive functional is given by

$$H(\theta, v) = \frac{1}{\theta} \log E [\exp(\theta \theta_T)] = E(\theta_T) + \frac{\theta}{2} \text{Var}(\theta_T) + O(\theta^2).$$

Proof. We have $H(\theta, v) = \frac{1}{\theta} \log E [\exp(\theta \theta_T)]$, by using the Taylor's expansion of the functions "exp" and "log" respectively, we get

$$\begin{aligned} H(\theta, v) &= \frac{1}{\theta} \log E [\exp(\theta \theta_T)] \\ &= \frac{1}{\theta} \log E \left[1 + \theta \theta_T + \frac{(\theta \theta_T)^2}{2!} + o[(\theta \theta_T)^2] \right] \\ &= \frac{1}{\theta} \log E \left[1 + \theta \theta_T + \frac{(\theta \theta_T)^2}{2!} + o[(\theta \theta_T)^2] \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\theta} \left[E(\theta \theta_T) + E \left(\frac{(\theta \theta_T)^2}{2!} \right) + E(o[(\theta \theta_T)^2]) \right] \\
&\quad - \frac{1}{\theta} \frac{\left[E(\theta \theta_T) + E \left(\frac{(\theta \theta_T)^2}{2!} \right) + E(o[(\theta \theta_T)^2]) \right]^2}{2} \\
&= E(\theta_T) + \frac{\theta}{2} \text{Var}(\theta_T) + O(\theta^2),
\end{aligned}$$

where $\text{Var}(\theta_T)$ denotes the variance of θ_T .

If $\theta < 0$, the $\text{Var}(\theta_T)$, as a measure of risk, improves the performance $H(\theta, v)$, in which case the optimizer is called risk seeker. But, when $\theta > 0$, the $\text{Var}(\theta_T)$ worsens the performance $H(\theta, v)$, in which case the optimizer is called risk averse. The risk-neutral functional $E(\theta_T)$ can be seen as a limit of the risk-sensitive functional $H(\theta, v)$ when $\theta \rightarrow 0$.

In what next let us introduce the following notations

Notation 3.1 Throughout this paper, we must use the following notations for every function $\rho \in \{b, \sigma, l\}$

$$\begin{cases} \varrho \xi(t) &= \varrho(t, t, x_s^v, v_s), \\ \varrho \zeta(t) &= \frac{\partial}{\partial \zeta} \varrho(t, t, x_s^v, v_s), \end{cases}$$

and

$$\begin{cases} \tilde{H}^\theta(t) &= \tilde{H}^\theta(t, \xi_t^v, x_t^v, v_t, \tilde{p}(t), \tilde{q}(t, t)), \\ \tilde{H}_\zeta^\theta(t) &= \frac{\partial}{\partial \zeta} \tilde{H}^\theta(t, \xi_t^v, x_t^v, v_t, \tilde{p}(t), \tilde{q}(t, t)), \end{cases}$$

such that $\tilde{H}^\theta$ represent the risk-neutral Hamiltonian, and H^θ represent the risk-sensitive Hamiltonian, is given by

$$\begin{cases} H^\theta(t) &= H^\theta(t, x_t^v, v_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t), D_2(t)), \\ H_\zeta^\theta(t) &= \frac{\partial}{\partial \zeta} H^\theta(t, x_t^v, v_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t), D_2(t)), \end{cases}$$

where $\zeta = x, v$ and v_t is an admissible control from U .

Lemma 3.2 We assume that assumptions 2.1 – 2.12.2 hold, Then there exist a unique F_t –adapted of processes (p_1, q_1) and (p_2, q_2) which solve the following system matrix of two forward SDVEs

$$\begin{cases} \tilde{p}(t) = ([c]lp_1(T)p_2(T)) \\ \quad + \int_t^T \left[\begin{pmatrix} 0 & 0 \\ \frac{\partial l}{\partial x}(s, x_s, v) & \frac{\partial b}{\partial x}(s, x_s, v) \end{pmatrix} \begin{pmatrix} p_1(s) \\ p_2(s) \end{pmatrix} \right] \\ \quad + \int_s^T \left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 b}{\partial z \partial x}(z, t, x_t, v_t) \end{pmatrix} \begin{pmatrix} p_1(z) \\ p_2(z) \end{pmatrix} \right] dz \\ \quad + \left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial \sigma}{\partial x}(s, s, x_s, v_s) \end{pmatrix} \begin{pmatrix} q_1(s, s) \\ q_2(s, s) \end{pmatrix} \right] \\ \quad + \int_s^T \left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 \sigma}{\partial z \partial x}(z, t, x_t, v_t) \end{pmatrix} \begin{pmatrix} q_1(z, t) \\ q_2(z, t) \end{pmatrix} \right] dz \Big] ds \\ \quad - \begin{pmatrix} q_1(t, s) \\ q_2(t, s) \end{pmatrix} dW_s \\ ([c]lp_1(T)p_2(T)) = \theta A_T^\theta([c]c1\Psi_x(x_T^u)), \end{cases} \quad (10)$$

With

$$\mathbb{E}\left[\sum_{i=1}^2 \sup_{t \in [0, T]} |p_i(t)|^2 + \sum_{i=1}^2 \int_0^T |q_i(s, t)|^2 dt\right] < \infty.$$

Proof. The adjoint processes associated with the system of two forward SDVEs (??), we have

$$\begin{cases} p_1(t) &= \theta A_T^\theta + \int_t^T \frac{\partial \tilde{H}^\theta}{\partial \xi}(s) ds - \int_t^T q_1(t, s) dW_s, \\ p_2(t) &= \theta A_T^\theta \Psi_x(x_T^u) + \int_t^T \frac{\partial \tilde{H}^\theta}{\partial x}(s) ds - \int_t^T q_2(t, s) dW_s. \end{cases} \quad (11)$$

We suppose here that $\tilde{H}^\theta$ be the Hamiltonian associated with the optimal state dynamics x^u , and the pair of adjoint processes $(\vec{p}(t), \vec{q}(t, t))$ given by

$$\begin{aligned} \tilde{H}^\theta(t) &:= \tilde{H}^\theta(t, \xi_t^v, x_t^v, v_t, \vec{p}(t), \vec{q}(t, t)) \\ &= \tilde{H}_0^\theta(t, \xi_t^v, x_t^v, v_t, \vec{p}(t), \vec{q}(t, t)) + \tilde{H}_1^\theta(t, \xi_t^v, x_t^v, v_t, \vec{p}(t), \vec{q}(t, t)). \end{aligned} \quad (12)$$

where

$$\begin{aligned} &\tilde{H}_0^\theta(t, \xi_t^v, x_t^v, v_t, \vec{p}(t), \vec{q}(t, t)) \\ &= l(t, x_t^v, v_t) p_1(t) + b(t, t, x_t^v, v_t) p_2(t) + \sigma(t, t, x_t^v, v_t) q_2(t, t), \end{aligned}$$

and

$$\begin{aligned} &\tilde{H}_1^\theta(t, \xi_t^v, x_t^v, v_t, \vec{p}(t), \vec{q}(t, t)) \\ &= \int_t^T \frac{\partial b}{\partial s}(s, t, x_t^v, v_t) p_2(s) ds + \int_t^T \frac{\partial \sigma}{\partial s}(s, t, x_t^v, v_t) q_2(s, t) ds. \end{aligned}$$

By replacing all the derivatives of the Hamiltonian functional $\tilde{H}^\theta$ with respect to all their components ξ and x , in the adjoint equations (11), we get

$$p_1(t) = \theta A_T^\theta - q_1(t, s) dW_s,$$

$$p_2(t) = \theta A_T^\theta \Psi_x(x_T^u)$$

$$\begin{aligned} &+ \int_t^T \left[\frac{\partial l}{\partial x}(s, x_s, u_s) p_1(s) + \frac{\partial b}{\partial x}(s, s, x_s, u_s) p_2(s) + \int_s^T \frac{\partial^2 b}{\partial z \partial x}(z, t, x_t, u_t) p_2(z) dz \right] ds \\ &+ \int_t^T \left[\frac{\partial \sigma}{\partial x}(s, s, x_s, u_s) q_2(s, s) + \int_s^T \frac{\partial^2 \sigma}{\partial z \partial x}(z, t, x_t, u_t) q_2(z, t) dz \right] ds \\ &- \int_t^T q_2(t, s) dW_s \\ &= \theta A_T^\theta \Psi_x(x_T^u) \\ &+ \int_t^T \left[\frac{\partial l}{\partial x}(s, x_s, u_s) p_1(s) + \frac{\partial b}{\partial x}(s, s, x_s, u_s) p_2(s) + (s - t) \frac{\partial^2 b}{\partial s \partial x}(s, t, x_t, u_t) p_2(s) \right] ds \end{aligned}$$

$$+ \int_t^T \left[\frac{\partial \sigma}{\partial x}(s, s, x_s, u_s) q_2(s, s) + (s - t) \frac{\partial^2 \sigma}{\partial s \partial x}(s, t, x_t, u_t) q_2(s, t) \right] ds$$

$$- \int_t^T q_2(t, s) dW_s.$$

Other written, as following

$$\left\{ \begin{array}{l} \vec{p}(t) = \begin{pmatrix} p_1(T) \\ p_2(T) \end{pmatrix} \\ + \int_t^T \left[\begin{pmatrix} 0 & 0 \\ \frac{\partial l}{\partial x}(s, x_s, u_s) & \frac{\partial b}{\partial x}(s, s, x_s, u_s) \end{pmatrix} \begin{pmatrix} p_1(s) \\ p_2(s) \end{pmatrix} \right] \\ + \int_s^T \left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 b}{\partial z \partial x}(z, t, x_t, u_t) \end{pmatrix} \begin{pmatrix} p_1(z) \\ p_2(z) \end{pmatrix} \right] dz \\ + \left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial \sigma}{\partial x}(s, s, x_s, u_s) \end{pmatrix} \begin{pmatrix} q_1(s, s) \\ q_2(s, s) \end{pmatrix} \right] \\ + \int_s^T \left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 \sigma}{\partial z \partial x}(z, t, x_t, u_t) \end{pmatrix} \begin{pmatrix} q_1(z, t) \\ q_2(z, t) \end{pmatrix} \right] dz \Big] ds \\ - \int_t^T \begin{pmatrix} q_1(t, s) \\ q_2(t, s) \end{pmatrix} dW_s, \\ \begin{pmatrix} p_1(T) \\ p_2(T) \end{pmatrix} = \theta A_T^\theta \begin{pmatrix} 1 \\ \Psi_x(x_T^u) \end{pmatrix}, \end{array} \right. \quad (13)$$

By differentiability form, we get

$$\left\{ \begin{array}{l} d_p^\rightarrow(t) = - \left[\begin{pmatrix} 0 & 0 \\ \frac{\partial l}{\partial x}(t, x_t, u_t) & \frac{\partial b}{\partial x}(t, t, x_t, u_t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} \right] dt \\ - \int_t^T \left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 b}{\partial s \partial x}(s, t, x_t, u_t) \end{pmatrix} \begin{pmatrix} p_1(s) \\ p_2(s) \end{pmatrix} \right] ds \\ - \left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial \sigma}{\partial x}(t, t, x_t, u_t) \end{pmatrix} \begin{pmatrix} q_1(t, t) \\ q_2(t, t) \end{pmatrix} \right] dt \\ - \int_t^T \left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 \sigma}{\partial s \partial x}(s, t, x_t, u_t) \end{pmatrix} \begin{pmatrix} q_1(s, t) \\ q_2(s, t) \end{pmatrix} \right] ds \Big] dt \\ + \left[\int_t^T \begin{pmatrix} \frac{\partial q_1}{\partial t}(t, s) \\ \frac{\partial q_2}{\partial t}(t, s) \end{pmatrix} dW_s \right] dt + \begin{pmatrix} q_1(t, t) \\ q_2(t, t) \end{pmatrix} dW_t, \\ \begin{pmatrix} p_1(T) \\ p_2(T) \end{pmatrix} = \theta A_T^\theta \begin{pmatrix} 1 \\ \Psi_x(x_T^u) \end{pmatrix}, \end{array} \right.$$

The following theorem is the stochastic maximum principle for the risk-neutral of two forward SDVE.

Theorem 3.1 We suppose that assumptions 2.1 – 2.12.2 hold. If u is an optimal control and (ξ^u, x^u) be the solution of (9) associated with u . Then, there exist a unique two pairs of F_t – adapted processes (p_1, q_1) and (p_2, q_2) verified (13), such that

$$H_v^\theta(t, \xi_t^u, x_t^u, u_t, \vec{p}(t), \vec{q}(t, t))(u_t - v_t) \leq 0, \quad (14)$$

for all $u \in U$, almost every t and P –almost surely.

Proof. The proof is based on the standard variational approach. First, we consider an admissible variation of the control with $v \in U$. Assumptions 2.1 – 2.12.2 guarantee the well-posedness of the state equation and the existence of the corresponding first-order variational processes.

Next, using the differentiability and boundedness conditions imposed on the coefficients, we derive the variational equations and obtain the first-order expansion of the cost functional.

Then, introducing the adjoint processes (p_1, q_1) and (p_2, q_2) satisfying (??), and applying the duality relation together with Itô's formula, the variation of the state variables is transferred to the Hamiltonian functional.

Consequently, the first variation of the performance functional can be expressed in terms of the derivative of the Hamiltonian with respect to the control variable.

Finally, since u is an optimal control, the variational inequality holds for every admissible control v , which yields condition (14). The existence and uniqueness of the adjoint processes follow from standard results for backward stochastic Volterra integral equations.

Detailed technical arguments can be found in [3, 18].

3.1. Derivatives of the Transformed Adjoint Equation and their Importance

We follow the new approach which has been used in [6; 7; 9; 10], and suggest a transformation of the adjoint processes (p_1, q_1) and (p_2, q_2) , such that we want to eliminate (p_1, q_1) in (13), and obtain the stochastic maximum principle with respect only by the new adjoint processes denoted by $(\tilde{p}_2, \tilde{q}_2)$. which are the adjoint equations associated with our initial control problem {(6), (7), (8)}. For this reason and according to the system (13), we have

$$\begin{cases} dp_1(t) &= \left[\int_t^T \frac{\partial q_1}{\partial t}(t, s) dW_s \right] dt + q_1(t, t) dW_t, \\ p_1(T) &= \theta A_T^\theta. \end{cases}$$

The explicit solution of this backward SDE given by

$$p_1(t) = \theta E[A_T^\theta | F_t] = \theta V_t^\theta, \quad (15)$$

where

$$V_t^\theta := E[A_T^\theta | F_t], 0 \leq t \leq T.$$

We choose the transformation from $(\vec{p}, \vec{q})$ to $(\tilde{p}, \tilde{q})$, with

$$\tilde{p}(t) = \begin{pmatrix} \tilde{p}_1(t) \\ \tilde{p}_2(t) \end{pmatrix} := \frac{1}{\theta V_t^\theta} \vec{p}(t), \quad 0 \leq t \leq T. \quad (16)$$

Then $\tilde{p}_1(t) = \frac{1}{\theta V_t^\theta} p_1(t) = 1$.

By using (13) and (16), we have

$$\tilde{p}(\cdot) := \begin{pmatrix} \tilde{p}_1(T) \\ \tilde{p}_2(T) \end{pmatrix} = \begin{pmatrix} 1 \\ \Psi_x(x_T^u) \end{pmatrix}.$$

The following properties of the generic martingale V^θ are essential to investigate the properties of these new processes $(\tilde{p}(t), \tilde{q}(t, t))$.

Lemma 3.3 Suppose the assumption (2.2) hold. Then the process

$$V^\theta(t) = \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right),$$

is uniformly bounded.

Proof. From the assumption (2.2), we know that the function Ψ and l are bounded by $C > 0$, we have

$$-\theta C(1+T) \leq \theta \left(\Psi(x_T^y) + \int_0^T l(s) ds \right) \leq \theta C(1+T).$$

By passing to the exponential, and the conditional expectation with respect to the filtration $(F_t)_{t \geq 0}$, we get

$$0 < \exp(-\theta C(1+T)) \leq V^\theta(t) \leq \exp(\theta C(1+T)),$$

which means the boundness of V^θ .

In this part, we want to prove the relationship between the exponential utility and the quadratic backward stochastic equation.

Based on the paper [11] Under the assumption (2.2), the process $(\Lambda_t^\theta, D(t))$ is the unique solution of the following quadratic backward SDE

$$\begin{cases} d\Lambda_t^\theta &= -\left[l(t) + \frac{\theta}{2}|D(t)|^2\right]dt + D(t)dW_t, \\ \Lambda_T^\theta &= \Psi(x_T^y), \end{cases} \quad (17)$$

where

$$E\left[\int_0^T |D(t)|^2 dt\right] < \infty.$$

The first component Λ_t^θ verified the following expression

$$\exp(\theta \Lambda_t^\theta) = E[A_{t,T}^\theta | F_t] = E\left[\exp\left(\theta \Psi(x_T^y) + \theta \int_t^T l(s) ds\right) | F_t\right]. \quad (18)$$

Lemma 3.4 The expressions (17) and (18) are equivalent.

Proof. We assume that (18) is true, it means that

$$\exp(\theta \Lambda_t^\theta) = E\left[\exp\left(\theta \Psi(x_T^y) + \theta \int_t^T l(s) ds\right) | F_t\right].$$

Multiplying the both sides of the above expression by $\exp\left(\theta \int_0^t l(s) ds\right)$, we get

$$\begin{aligned} \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right) &= E\left[\exp\left(\theta \Psi(x_T^y) + \theta \int_t^T l(s) ds + \theta \int_0^t l(s) ds\right) | F_t\right] \\ &= E\left[\exp\left(\theta \Psi(x_T^y) + \theta \int_0^T l(s) ds\right) | F_t\right] \\ &= E[A_T^\theta | F_t]. \end{aligned}$$

If $t = 0$, $V^\theta(0) = E[A_T^\theta | F_0] = E[A_T^\theta] = \exp(\theta \Lambda_0^\theta)$.

$E[A_T^\theta | F_t]$ is a square integrable F_t -martingale, by Itô's martingale representation theorem there exists a unique

F_t –progressively measurable process $\phi(t)$ with respect to norm $\|Q\|_{M^2([0,T],R)}$ such that

$$E[A_T^\theta | F_t] = E[A_T^\theta] + \theta \int_0^t \phi(s) dW_s = \exp(\theta \Lambda_0^\theta) + \theta \int_0^t \phi(s) dW_s.$$

Hence

$$\exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right) - \exp(\theta \Lambda_0^\theta) = \theta \int_0^t \phi(s) dW_s.$$

We apply Itô's formula to $\exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right)$, we find

$$\begin{aligned} & d\left(\exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right)\right) \\ &= \theta l(t) \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right) dt + \theta \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right) d\Lambda_t^\theta \\ &+ \frac{\theta^2}{2} \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right) \langle d\Lambda_t^\theta, d\Lambda_t^\theta \rangle \\ &= \theta \phi(t) dW_t, \end{aligned}$$

where $\langle ., . \rangle$ denotes the quad ratic variation. Then

$$l(t)dt + d\Lambda_t^\theta + \frac{\theta}{2} \langle d\Lambda_t^\theta, d\Lambda_t^\theta \rangle = \phi(t) \exp\left(-\theta \Lambda_t^\theta - \theta \int_0^t l(s) ds\right) dW_t. \quad (19)$$

Hence,

$$\langle d\Lambda_t^\theta, d\Lambda_t^\theta \rangle = \left[\phi(t) \exp\left(-\theta \Lambda_t^\theta - \theta \int_0^t l(s) ds\right) \right]^2 dt = |D(t)|^2 dt. \quad (20)$$

Then, by replacing (20) in (19), we have the backward quadratic as the following expression

$$\begin{cases} d\Lambda_t^\theta &= -\left[l(t) + \frac{\theta}{2}|D(t)|^2\right] dt + D(t) dW_t, \\ \Lambda_T^\theta &= \Psi(x_T^u), \end{cases}$$

where

$$D(t) = \phi(t) \exp\left(-\theta \Lambda_t^\theta - \theta \int_0^t l(s) ds\right).$$

In the other hand, we assume that quadratic backward SDE (17) is hold, then by applying the Itô's formula to $\exp(\theta \Lambda_t^\theta)$, we have

$$d(\exp(\theta \Lambda_t^\theta)) = \theta \exp(\theta \Lambda_t^\theta) d\Lambda_t^\theta + \frac{\theta^2}{2} \exp(\theta \Lambda_t^\theta) |D(t)|^2 dt.$$

Then

$$d(\exp(\theta \Lambda_t^\theta)) = -\theta l(t) \exp(\theta \Lambda_t^\theta) dt - \frac{\theta^2}{2} \exp(\theta \Lambda_t^\theta) |D(t)|^2 dt$$

$$+ \theta \exp(\theta \Lambda_t^\theta) D(t) dW_t + \frac{\theta^2}{2} \exp(\theta \Lambda_t^\theta) |D(t)|^2 dt.$$

Finally, we obtain

$$d(\exp(\theta \Lambda_t^\theta)) + \theta l(t) \exp(\theta \Lambda_t^\theta) dt = \theta D(t) \exp(\theta \Lambda_t^\theta) dW_t.$$

Multiplying the both sides of the above expression by $\exp\left(\theta \int_0^t l(s) ds\right)$, we get

$$\begin{aligned} \exp\left(\theta \int_0^t l(s) ds\right) d(\exp(\theta \Lambda_t^\theta)) + \theta l(t) \exp\left(\theta \int_0^t l(s) ds\right) \exp(\theta \Lambda_t^\theta) dt \\ = \theta D(t) \exp\left(\theta \int_0^t l(s) ds\right) \exp(\theta \Lambda_t^\theta) dW_t. \end{aligned}$$

The right side is the same as the $d\left(\exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right)\right)$, then we have

$$d\left(\exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right)\right) = \theta D(t) \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right) dW_t.$$

By passing to integral from t into T , we find

$$\int_t^T d\left(\exp\left(\theta \Lambda_s^\theta + \theta \int_0^s l(r) dr\right)\right) = \theta \int_t^T D(s) \exp\left(\theta \Lambda_s^\theta + \theta \int_0^s l(r) dr\right) dW_s.$$

Then

$$\begin{aligned} \exp\left(\theta \Lambda_T^\theta + \theta \int_0^T l(r) dr\right) &= \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(r) dr\right) \\ &+ \theta \int_t^T D(s) \exp\left(\theta \Lambda_s^\theta + \theta \int_0^s l(r) dr\right) dW_s. \end{aligned}$$

Then, by using the definition of the conditional expectation in the both sides of the last equality, we obtain

$$\begin{aligned} E\left[\exp\left(\theta \Lambda_T^\theta + \theta \int_0^T l(r) dr\right) | F_t\right] \\ = E\left[\exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(r) dr\right) | F_t\right] \\ + \theta E\left[\int_t^T D(s) \exp\left(\theta \Lambda_s^\theta + \theta \int_0^s l(r) dr\right) dW_s | F_t\right]. \end{aligned}$$

In according to the martingale of Itô's integral, and the measurability with respect to the filtration F_t , we get

$$E\left[\int_t^T D(s) \exp\left(\theta \Lambda_s^\theta + \theta \int_0^s l(r) dr\right) dW_s | F_t\right] = 0.$$

Then

$$E \left[\exp \left(\theta \Lambda_T^\theta + \int_0^T l(r) dr \right) | F_t \right] = \exp \left(\theta \Lambda_t^\theta + \int_0^t l(r) dr \right).$$

From the definition of the terminal condition of the quadratic backward SDE (17), we now that $\Lambda_T^\theta = \Psi(x_T^u)$, then

$$E \left[\exp \left(\theta \Psi(x_T^u) + \int_0^T l(r) dr \right) | F_t \right] = \exp \left(\theta \Lambda_t^\theta + \int_0^t l(r) dr \right).$$

Then

$$E \left[\exp \left(\theta \Psi(x_T^u) + \int_t^T l(s) ds \right) | F_t \right] = \exp(\theta \Lambda_t^\theta).$$

This leads to the expression (17).

Lemma 3.5 The martingale V_t^θ can be rewritten in exponential as the following form

$$V_t^\theta = \exp \left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds \right).$$

Proof. We know that

$$\begin{aligned} V_t^\theta &= E \left[\exp \left(\theta \Psi(x_T^u) + \theta \int_0^T l(s) ds \right) | F_t \right] \\ &= \exp \left(\theta \int_0^t l(s) ds \right) \times E \left[\exp \left(\theta \Psi(x_T^u) + \theta \int_t^T l(s) ds \right) | F_t \right]. \end{aligned}$$

From the Lemma (3.4), the equation (18) can be transformed into

$$\begin{aligned} V_t^\theta &= E \left[\exp \left(\theta \Psi(x_T^u) + \theta \int_0^T l(s) ds \right) | F_t \right] \\ &= \exp \left(\theta \int_0^t l(s) ds \right) \times \exp(\theta \Lambda_t^\theta) \\ &= \exp \left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds \right). \end{aligned}$$

Then, the lemma is proved.

Remark 3.1 We can easily prove from (17) and the boundness of l that

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\Lambda_t^\theta| \right] \leq C_T, \quad (21)$$

where C_T is a positive constant depends only to T .

Lemma 3.6 The martingale V_t^θ is the solution of the following backward SDE

$$\begin{cases} dV_t^\theta &= \theta V_t^\theta D(t) dW_t, \\ V_T^\theta &= A_T^\theta. \end{cases} \quad (22)$$

Beside, the process L_t^θ defines on $(\Omega, F, (F_t)_{t \geq 0}, P)$ well given by

$$L_t^\theta := \frac{V_t^\theta}{V_0^\theta} = \exp\left(\theta \int_0^t D(s) dW_s - \frac{\theta^2}{2} \int_0^t |D(s)|^2 ds\right), \forall 0 \leq t \leq T, \quad (23)$$

is a uniformly bounded F_t –martingale.

Proof. In order to establish (22), Itô's formula is applied to the process

$$V_t^\theta = \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right).$$

Then

$$\begin{aligned} dV_t^\theta &= d\left(\exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right)\right) \\ &= \theta l(t) \left(\exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right)\right) dt + \theta \left(\exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right)\right) d\Lambda_t^\theta \\ &\quad + \frac{\theta^2}{2} |D(t)|^2 \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right) dt \\ &= \theta l(t) \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right) dt \\ &\quad + \theta \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right) \left[-\left[l(t) + \frac{\theta}{2} |D(t)|^2\right] dt + D(t) dW_t\right] \\ &\quad + \frac{\theta^2}{2} |D(t)|^2 \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right) dt \\ &= \theta \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right) D(t) dW_t \\ &= \theta V_t^\theta D(t) dW_t. \end{aligned}$$

Now we want to prove (23), from (17) we get

$$\Lambda_t^\theta = \Lambda_0^\theta - \int_0^t l(s) ds - \frac{\theta}{2} \int_0^t |D(s)|^2 ds + \int_0^t D(s) dW_s.$$

Then

$$\begin{aligned} V_t^\theta &= \exp\left(\theta \Lambda_t^\theta + \theta \int_0^t l(s) ds\right) \\ &= \exp\left(\theta \Lambda_0^\theta - \theta \int_0^t l(s) ds - \frac{\theta^2}{2} \int_0^t |D(s)|^2 ds + \theta \int_0^t D(s) dW_s + \theta \int_0^t l(s) ds\right) \\ &= \exp\left(\theta \Lambda_0^\theta - \frac{\theta^2}{2} \int_0^t |D(s)|^2 ds + \theta \int_0^t D(s) dW_s\right). \end{aligned}$$

We divide on V_0^θ such that $V_0^\theta = \exp(\theta \Lambda_0^\theta)$, we get

$$L_t^\theta := \frac{V_t^\theta}{V_0^\theta} = \exp\left(\theta \int_0^t D(s) dW_s - \frac{\theta^2}{2} \int_0^t |D(s)|^2 ds\right), \forall 0 \leq t \leq T,$$

L_t^θ is called geometric Brownian motion, L_t^θ is therefore a uniformly bounded F_t –martingale.

In the next, we will state and prove the necessary optimality conditions for the system driven by forward SDVE with a risk-sensitive performance cost type.

Proposition 3.1 The main risk-sensitive second adjoint equation for $(\tilde{p}_2, \tilde{q}_2)$ and (V^θ, D) become

$$\begin{cases} d\tilde{p}_2(t) &= -H_x^\theta(t)dt + [\tilde{q}_2(t, t) - \theta D_2(t, t)\tilde{p}_2(t)]dW_t^\theta, \\ dV_t^\theta &= \theta V_t^\theta D(t)dW_t, \\ \tilde{p}_2(T) &= \Psi_x(x_T^u), V_T^\theta = A_T^\theta. \end{cases} \quad (24)$$

The solution $(\tilde{p}, \tilde{q}, V^\theta, D)$ of the system (24) is unique, such that

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |\tilde{p}(t)|^2 + \sup_{t \in [0, T]} |V^\theta(t)|^2 + \int_0^T |D(t)|^2 dt \right] &< \infty, \\ E \left[\int_0^T \int_0^T |\tilde{q}(t, s)|^2 ds dt + \int_0^T \int_0^T \left| \frac{\partial \tilde{q}(t, s)}{\partial t} \right|^2 ds dt \right] &< \infty, \end{aligned} \quad (25)$$

with

$$\begin{aligned} H^\theta(t) &:= H^\theta(t, x_t^v, v_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t), D_2(t)) \\ &= H_0^\theta(t, x_t^v, v_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t), D_2(t)) \\ &\quad + H_1^\theta(t, x_t^v, v_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t), D_2(t)), \end{aligned} \quad (26)$$

where

$$\begin{aligned} H_0^\theta(t, x_t^v, v_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t), D_2(t)) \\ = l(t, x_t^v, v_t) + b(t, t, x_t^v, v_t)\tilde{p}_2(t) + \sigma(t, t, x_t^v, v_t)\tilde{q}_2(t, t), \end{aligned}$$

and

$$\begin{aligned} H_1^\theta(t, x_t^v, v_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t), D_2(t)) \\ = \int_t^T \frac{\partial b}{\partial s}(t, s, x_s^v, v_s)\tilde{p}_2(s)ds + \int_t^T \frac{\partial \sigma}{\partial s}(t, s, x_s^v, v_s)\tilde{q}_2(s, t)ds. \end{aligned}$$

Proof. We want to identify the processes $\tilde{\alpha}$ and $\tilde{\beta}$ such that

$$d\tilde{p}(t) = \tilde{\alpha}(t)dt + \tilde{\beta}(t)dW_t. \quad (27)$$

We applying Itô's formula to the processes $\vec{p}(t) = \theta V_t^\theta \tilde{p}(t)$,

$$d\overleftarrow{p}(t) = \theta V_t^\theta d\vec{p}(t) + \theta \tilde{p}(t)dV_t^\theta + \theta \langle dV_t^\theta, d\tilde{p}(t) \rangle.$$

Then

$$d\tilde{p}(t) = \frac{1}{\theta V_t^\theta} d\vec{p}(t) - \frac{1}{\theta V_t^\theta} \theta \tilde{p}(t)dV_t^\theta - \frac{1}{\theta V_t^\theta} \theta \langle dV_t^\theta, d\tilde{p}(t) \rangle.$$

By using the expressions of V_t^θ and $\tilde{p}(t)$, we obtain

$$dV_t^\theta = \theta V_t^\theta D(t) dW_t, \quad \langle dV_t^\theta, d\tilde{p}(t) \rangle = \theta V_t^\theta D(t) \tilde{\beta}(t) dt.$$

Then, we find

$$\begin{aligned} d\tilde{p}(t) &= \frac{1}{\theta V_t^\theta} d\vec{p}(t) - \frac{1}{\theta V_t^\theta} \theta \tilde{p}(t) dV_t^\theta - \frac{1}{\theta V_t^\theta} \theta \langle dV_t^\theta, d\tilde{p}(t) \rangle \\ &= \frac{1}{\theta V_t^\theta} d\vec{p}(t) - \frac{1}{\theta V_t^\theta} \theta \tilde{p}(t) [\theta V_t^\theta D(t) dW_t] - \frac{1}{\theta V_t^\theta} \theta [\theta V_t^\theta D(t) \tilde{\beta}(t) dt] \\ &= \frac{1}{\theta V_t^\theta} d\vec{p}(t) - \theta \tilde{p}(t) D(t) dW_t - \theta D(t) \tilde{\beta}(t) dt. \end{aligned}$$

We want to determine the coefficients $\tilde{\alpha}(t)$ and $\tilde{\beta}(t)$ such that

$$\begin{aligned} d\tilde{p}(t) &= - \begin{pmatrix} 0 & 0 \\ \frac{\partial l}{\partial x}(t, x_t, u_t) & \frac{\partial b}{\partial x}(t, x_t, u_t) \end{pmatrix} \begin{pmatrix} \tilde{p}_1(t) \\ \tilde{p}_2(t) \end{pmatrix} dt \\ &\quad - \left[\int_t^T \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 b}{\partial s \partial x}(s, t, x_t, u_t) \end{pmatrix} \begin{pmatrix} \tilde{p}_1(s) \\ \tilde{p}_2(s) \end{pmatrix} ds \right] dt \\ &\quad - \left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial \sigma}{\partial x}(t, t, x_t, u_t) \end{pmatrix} \begin{pmatrix} \tilde{q}_1(t, t) \\ \tilde{q}_2(t, t) \end{pmatrix} \right] dt \\ &\quad - \left[\int_t^T \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 \sigma}{\partial s \partial x}(s, t, x_t, u_t) \end{pmatrix} \begin{pmatrix} \tilde{q}_1(s, t) \\ \tilde{q}_2(s, t) \end{pmatrix} ds \right] dt \\ &\quad + \left[\int_t^T \begin{pmatrix} \frac{\partial \tilde{q}_1}{\partial t}(t, s) \\ \frac{\partial \tilde{q}_2}{\partial t}(t, s) \end{pmatrix} dW_s \right] dt - \left[\theta \begin{pmatrix} D_1(t) \\ D_2(t) \end{pmatrix} \tilde{\beta}(t) \right] dt \\ &\quad + \left[\begin{pmatrix} \tilde{q}_1(t, t) \\ \tilde{q}_2(t, t) \end{pmatrix} - \theta \begin{pmatrix} D_1(t, t) \\ D_2(t, t) \end{pmatrix} \tilde{p}(t) \right] dW_t. \end{aligned}$$

By identifying the coefficients the above equation and equation (27), the diffusion coefficient $\tilde{\beta}(t)$ it will be written as

$$\tilde{\beta}(t) = \begin{pmatrix} \tilde{q}_1(t, t) \\ \tilde{q}_2(t, t) \end{pmatrix} - \theta \begin{pmatrix} D_1(t, t) \\ D_2(t, t) \end{pmatrix} \tilde{p}(t), \quad (28)$$

the drift term $\tilde{\alpha}(t)$ of the process $\tilde{p}(t)$

$$\begin{aligned} \tilde{\alpha}(t) &= - \begin{pmatrix} 0 & 0 \\ \frac{\partial l}{\partial x}(t, x_t, u_t) & \frac{\partial b}{\partial x}(t, x_t, u_t) \end{pmatrix} \begin{pmatrix} \tilde{p}_1(t) \\ \tilde{p}_2(t) \end{pmatrix} \\ &\quad - \int_t^T \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 b}{\partial s \partial x}(s, t, x_t, u_t) \end{pmatrix} \begin{pmatrix} \tilde{p}_1(s) \\ \tilde{p}_2(s) \end{pmatrix} ds \\ &\quad - \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial \sigma}{\partial x}(t, t, x_t, u_t) \end{pmatrix} \begin{pmatrix} \tilde{q}_1(t, t) \\ \tilde{q}_2(t, t) \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
& - \int_t^T \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 \sigma}{\partial s \partial x} \end{pmatrix} (s, t, x_t, u_t) \begin{pmatrix} \tilde{q}_1(s, t) \\ \tilde{q}_2(s, t) \end{pmatrix} ds \\
& + \int_t^T \begin{pmatrix} \frac{\partial \tilde{q}_1}{\partial t} \\ \frac{\partial \tilde{q}_2}{\partial t} \end{pmatrix} (t, s) dW_s - \theta \begin{pmatrix} D_1(t) \\ D_2(t) \end{pmatrix} \tilde{\beta}(t).
\end{aligned}$$

It is easily verified that

$$\begin{cases} d\tilde{p}_1(t) = \left[\int_t^T \frac{\partial \tilde{q}_1}{\partial t} (t, s) dW_s \right] dt + \tilde{\beta}_1(t) [dW_t - \theta D_1(t) dt], \\ \tilde{p}_1(T) = 1. \end{cases}$$

In view of (23), we may use Girsanov's Theorem (See [5], Theorem 3.1 pages 124 – 125), to claim that

$$d\tilde{p}_1(t) = \left[\int_t^T \frac{\partial \tilde{q}_1}{\partial t} (t, s) dW_s \right] dt + \tilde{\beta}_1(t) dW_t^\theta, \quad P^\theta - a.s., \quad \tilde{p}_1(T) = 1,$$

with,

$$dW_t^\theta = dW_t - \theta D(t) dt,$$

is a P^θ –Brownian motion, where,

$$\left. \frac{dP^\theta}{dP} \right|_{F_t} := L_t^\theta = \exp \left(\theta \int_0^t D(s) dW_s - \frac{\theta^2}{2} \int_0^t |D(s)|^2 ds \right), \quad \forall 0 \leq t \leq T.$$

The probability measures P^θ and P are in fact equivalent. Hence, noting that $\tilde{p}_1(t) := \frac{1}{\theta V_t^\theta} p_1(t)$ is square-integrable, we get that $\tilde{p}_1(t) = E^{P^\theta}[\tilde{p}_1(T) | F_t] = 1$. Thus, its quadratic variation $\int_0^T |\tilde{q}_1(t, t)|^2 dt = 0$. This implies that, for almost every $0 \leq t \leq T$, $\tilde{q}_1(t, t) = 0$, P^θ and $P - a.s.$ Then,

$$\begin{aligned}
d\tilde{p}(t) = & - \left[\begin{pmatrix} 0 & 0 \\ \frac{\partial l}{\partial x} & \frac{\partial b}{\partial x} \end{pmatrix} (t, t, x_t, v_t) \begin{pmatrix} \tilde{p}_1(t) \\ \tilde{p}_2(t) \end{pmatrix} \right] dt \\
& - \left[\int_t^T \left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 b}{\partial s \partial x} \end{pmatrix} (s, t, x_t, v_t) \begin{pmatrix} \tilde{p}_1(s) \\ \tilde{p}_2(s) \end{pmatrix} \right] ds \right] dt \\
& - \left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial \sigma}{\partial x} \end{pmatrix} (t, t, x_t, v_t) \begin{pmatrix} \tilde{q}_1(t, t) \\ \tilde{q}_2(t, t) \end{pmatrix} \right] dt \\
& - \left[\int_t^T \left[\begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 b}{\partial s \partial x} \end{pmatrix} (s, t, x_t, v_t) \begin{pmatrix} \tilde{q}_1(s, t) \\ \tilde{q}_2(s, t) \end{pmatrix} \right] ds \right] dt \\
& + \left[\int_t^T \begin{pmatrix} \frac{\partial \tilde{q}_1}{\partial t} \\ \frac{\partial \tilde{q}_2}{\partial t} \end{pmatrix} (t, s) dW_s \right] dt + \tilde{\beta}(t) dW_t^\theta.
\end{aligned} \tag{29}$$

Therefore, the second component of $\tilde{p}_2$ in (29), is given by

$$\begin{aligned}
d\tilde{p}_2(t) = & -\left[\frac{\partial l}{\partial x}(t, x_t, u_t) + \frac{\partial b}{\partial x}(t, t, x_t, u_t)\tilde{p}_2(t, t) \right. \\
& + \int_t^T \frac{\partial^2 b}{\partial s \partial x}(s, t, x_t, u_t)\tilde{p}_2(s)ds + \frac{\partial \sigma}{\partial x}(t, t, x_t, u_t)\tilde{p}_2(t) \\
& + \int_t^T \frac{\partial^2 \sigma}{\partial s \partial x}(s, t, x_t, u_t)\tilde{q}_2(s)ds - \int_t^T \frac{\partial \tilde{q}_2}{\partial t}(t, s) dW_s \Big] dt \\
& + [\tilde{q}_2(t, t) - \theta D_2(t, t)\tilde{p}_2(t)]dW_t^\theta.
\end{aligned}$$

The main risk-sensitive second adjoint equation for $(\tilde{p}_2, \tilde{q}_2)$ and (V^θ, D) becomes

$$\begin{cases} d\tilde{p}_2(t) &= -H_x^\theta(t)dt + [\tilde{q}_2(t, t) - \theta D_2(t, t)\tilde{p}_2(t)]dW_t^\theta \\ dV_t^\theta &= \theta V_t^\theta D(t)dW_t, \\ \tilde{p}_2(T) &= \Psi_x(x_T^u), V_T^\theta = A_T^\theta. \end{cases}$$

The solution $(\tilde{p}, \tilde{q}, V^\theta, D)$ of the system (24) is unique, such that

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\tilde{p}(t)|^2 + \sup_{t \in [0, T]} |V^\theta(t)|^2 + \int_0^T (|\tilde{q}(t, t)|^2 + |D(t)|^2) dt \right] < \infty,$$

where

$$\begin{aligned}
H^\theta(t) &:= H^\theta(t, x_t^u, u_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t), D_2(t)) \\
&= H_0^\theta(t, x_t^u, u_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t), D_2(t)) \\
&\quad + H_1^\theta(t, x_t^u, u_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t), D_2(t)).
\end{aligned}$$

where

$$\begin{aligned}
H_0^\theta(t, x_t^u, u_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t), D_2(t)) \\
= l(t, x_t^u, u_t) + b(t, t, x_t^u, u_t)\tilde{p}_2(t) + \sigma(t, t, x_t^u, u_t)\tilde{q}_2(t, t),
\end{aligned}$$

and

$$\begin{aligned}
H_1^\theta(t, x_t^u, u_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t), D_2(t)) \\
= \int_t^T \frac{\partial b}{\partial s}(t, s, x_s^u, u_s)\tilde{p}_2(s)ds + \int_t^T \frac{\partial \sigma}{\partial s}(t, s, x_s^u, u_s)\tilde{q}_2(s, t)ds.
\end{aligned}$$

The proof is completed.

Now, we can state the necessary optimality conditions

Theorem 3.2 (Necessary optimality conditions for risk-sensitive) We assume that Assumptions 2.1 – 2.12.2 hold. If (x^u, u) is an optimal solution of the risk-sensitive control problem $\{(6), (7), (8)\}$, then there exist F_t –adapted processes (V^θ, D) and $(\tilde{p}_2, \tilde{q}_2)$ that satisfy (24), (25), such that

$$H_v^\theta(t, x_t^u, u_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t), D_2(t))(u_t - v_t) \leq 0, \quad (30)$$

for all $u \in U$, almost every $0 \leq t \leq T$ and P –almost surely.

Proof. The Hamiltonian $\tilde{H}^\theta$ associated with (9), given by (12) satisfies

$$H^\theta(t, \xi_t^u, x_t^u, u_t, \tilde{p}(t), \tilde{q}(t, t)) = \{\theta V_t^\theta\} H^\theta(t, x_t^u, u_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t), D_2(t)), \quad (31)$$

and H^θ is the risk-sensitive Hamiltonian given by (26). To arrive at a risk-sensitive stochastic maximum principle expressed in terms of the adjoint processes $(\tilde{p}_2, \tilde{q}_2)$ and (V^θ, D) , which solve (24). Hence, since $V^\theta > 0$, the variational inequality (14) translates into

$$H_v^\theta(t, x_t^u, u_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t), D_2(t))(u_t - v_t) \leq 0,$$

for all $u \in U$, almost every $0 \leq t \leq T$ and P –almost surely. This completed the proof of Theorem 3.2.

3.2. Risk-Sensitive Sufficient Optimality Conditions

This section is concerned with a study of the necessary conditions of optimality (14) when it becomes sufficient.

For all $v \in U$, we consider the following SDVE system,

$$\begin{cases} dx_t^v &= b(t, t, x_t^v, v_t)dt + \sigma(t, t, x_t^v, v_t)dW_t \\ &+ \left(\int_0^t \frac{\partial b}{\partial t}(t, s, x_s^v, v_s)ds \right)dt + \left(\int_0^t \frac{\partial \sigma}{\partial t}(t, s, x_s^v, v_s)dW_s \right)dt, \\ x_0^v &= \alpha, \end{cases}$$

where $x_0^v = \alpha$ is one dimensional F_0 –measurable such that $E|\alpha|^2 < \infty$.

Theorem 3.3 (Sufficient optimality conditions for risk-sensitive) Assume that the function Ψ is convex with respect to x and the function

$\tilde{H}^\theta(t, \xi_t^v, x_t^v, v_t, \tilde{p}(t), \tilde{q}(t, t))$ is convex with respect to (x, v) . Then, u is an optimal solution of the control problem $\{(6), (7), (8)\}$ if it satisfies (14).

Proof. Let u be an arbitrary element of U (candidate to be optimal). For any $v \in U$, we have

$$\begin{aligned} J^\theta(v) - J^\theta(u) \\ = E[\exp(\theta\{\Psi(x_T^v) + \xi_T^v\})] - E[\exp(\theta\{\Psi(x_T^u) + \xi_T^u\})]. \end{aligned}$$

By applying the Taylor's expansion, and since Ψ is convex, we get

$$J^\theta(v) - J^\theta(u) \geq E[\theta A_T^\theta(\xi_T^v - \xi_T^u)] - E[\theta [A_T^\theta \Psi_x(x_T^u)](x_T^v - x_T^u)].$$

It follows from (10), we notice that $p_1(T) = \theta A_T^\theta$, and $p_2(T) = \theta [A_T^\theta \Psi_x(x_T^u)]$, then we have

$$J^\theta(v) - J^\theta(u) \geq E[p_1(T)(\xi_T^v - \xi_T^u)] + E[p_2(T)(x_T^v - x_T^u)]. \quad (32)$$

By applying Itô's formula to $p_1(t)(\xi_t^v - \xi_t^u)$, we get

$$d(p_1(t)(\xi_t^v - \xi_t^u)) = p_1(t)d(\xi_t^v - \xi_t^u) + (\xi_t^v - \xi_t^u)dp_1(t) + \langle dp_1(t), d(\xi_t^v - \xi_t^u) \rangle,$$

by integrating $\int_0^T$, and taking expectation, we get

$$E[p_1(T)(\xi_T^v - \xi_T^u)] = E\left[\int_0^T (l(t, x_t^v, v_t) - l(t, x_t^u, u_t))p_1(t)dt\right] \quad (33)$$

Now, by applying Itô's formula to $p_2(t)(x_t^v - x_t^u)$, we get

$$d(p_2(t)(x_t^v - x_t^u))$$

$$\begin{aligned}
&= p_2(t)d(x_t^v - x_t^u) + (x_t^v - x_t^u)dp_2(t) + \langle dp_2(t), d(x_t^v - x_t^u) \rangle \\
&= p_2(t)[b(t, t, x_t^v, v_t) - b(t, t, x_t^u, u_t)]dt + p_2(t)[\sigma(t, t, x_t^v, v_t) - \sigma(t, t, x_t^u, u_t)]dW_t \\
&+ p_2(t) \left(\int_0^t \frac{\partial b}{\partial t}(t, s, x_s^v, v_s)ds - \int_0^t \frac{\partial b}{\partial t}(t, s, x_s^u, u_s)ds \right) dt \\
&+ p_2(t) \left(\int_0^t \frac{\partial \sigma}{\partial t}(t, s, x_s^v, v_s) dW_s - \int_0^t \frac{\partial \sigma}{\partial t}(t, s, x_s^u, u_s) dW_s \right) dt \\
&- (x_t^v - x_t^u) \left[\frac{\partial l}{\partial x}(t, x_t^u, u_t)p_1(t) + \frac{\partial b}{\partial x}(t, t, x_t^u, u_t)p_2(t) \right] dt \\
&- (x_t^v - x_t^u) \left[\int_t^T \left(\frac{\partial^2 b}{\partial s \partial x}(s, t, x_t^u, u_t) \right) p_2(s) ds \right] dt \\
&- (x_t^v - x_t^u) \left[\frac{\partial \sigma}{\partial x}(t, t, x_t^u, u_t)q_2(t, t) + \int_t^T \frac{\partial^2 \sigma}{\partial s \partial x}(s, t, x_t^u, u_t)q_2(s, t)ds \right] dt \\
&+ (x_t^v - x_t^u) \left[\int_t^T \frac{\partial q_2}{\partial t}(t, s) dW_s \right] dt + (x_t^v - x_t^u)q_2(t, t)dW_t \\
&+ [\sigma(t, t, x_t^v, v_t) - \sigma(t, t, x_t^u, u_t)]q_2(t, t)dt.
\end{aligned}$$

by integrating $\int_0^T$, and taking expectation, we get

$$\begin{aligned}
&p_2(T)(x_T^v - x_T^u) \\
&= E \left[\int_0^T p_2(t)[b(t, t, x_t^v, v_t) - b(t, t, x_t^u, u_t)] dt \right] \\
&+ E \left[\int_0^T p_2(t) \left(\int_0^t \frac{\partial b}{\partial t}(t, s, x_s^v, v_s)ds - \int_0^t \frac{\partial b}{\partial t}(t, s, x_s^u, u_s)ds \right) dt \right] \\
&+ E \left[\int_0^T p_2(t) \left(\int_0^t \frac{\partial \sigma}{\partial t}(t, s, x_s^v, v_s)dW_s - \int_0^t \frac{\partial \sigma}{\partial t}(t, s, x_s^u, u_s)dW_s \right) dt \right] \\
&- E \left[\int_0^T (x_t^v - x_t^u) \left[\frac{\partial l}{\partial x}(t, x_t^u, u_t)p_1(t) + \frac{\partial b}{\partial x}(t, t, x_t^u, u_t)p_2(t) \right] dt \right] \\
&- E \left[\int_0^T (x_t^v - x_t^u) \left[\int_t^T \left(\frac{\partial^2 b}{\partial s \partial x}(s, t, x_t^u, u_t) \right) p_2(s) ds \right] dt \right] \\
&- E \left[\int_0^T (x_t^v - x_t^u) \left[\frac{\partial \sigma}{\partial x}(t, t, x_t^u, u_t)q_2(t, t) + \frac{\partial^2 \sigma}{\partial s \partial x}(s, t, x_t^u, u_t)(s, t)ds \right] dt \right] \\
&+ E[\sigma(t, t, x_t^v, v_t) - \sigma(t, t, x_t^u, u_t)]q_2(t, t)dt].
\end{aligned} \tag{34}$$

By the Fubini's theorem, we get

$$\begin{aligned}
&E \left[\int_0^T p_2(t) \left(\int_0^t \frac{\partial b}{\partial t}(t, s, x_s^v, v_s)ds - \int_0^t \frac{\partial b}{\partial t}(t, s, x_s^u, u_s)ds \right) dt \right] \\
&= E \left[\int_0^T \left(\int_s^T p_2(t) \frac{\partial b}{\partial t}(t, s, x_s^v, v_s)dt - \int_s^T p_2(t) \frac{\partial b}{\partial t}(t, s, x_s^u, u_s)dt \right) ds \right]
\end{aligned} \tag{35}$$

$$= E \left[\int_0^T \int_t^T p_2(s) \left(\frac{\partial b}{\partial s}(t, s, x_s^v, v_s) - \frac{\partial b}{\partial s}(t, s, x_s^u, u_s) \right) ds dt \right].$$

The generalized duality formula for the Brownian motion (2), yields

$$\begin{aligned} & E \left[p_2(t) \left(\frac{\partial \sigma}{\partial t}(t, s, x_s^v, v_s) - \frac{\partial \sigma}{\partial t}(t, s, x_s^u, u_s) \right) dW_s dt \right] \\ &= \int_0^T E \left[\int_0^t p_2(t) \left(\frac{\partial \sigma}{\partial t}(t, s, x_s^v, v_s) - \frac{\partial \sigma}{\partial t}(t, s, x_s^u, u_s) \right) dW_s \right] dt \\ &= \int_0^T E \left[\int_0^t E[D_s p_2(t) | F_s] \left(\frac{\partial \sigma}{\partial t}(t, s, x_s^v, v_s) - \frac{\partial \sigma}{\partial t}(t, s, x_s^u, u_s) \right) ds \right] dt \end{aligned}$$

Fubini's theorem gives

$$\begin{aligned} & E \left[\int_0^T p_2(t) \left(\int_0^t \frac{\partial \sigma}{\partial t}(t, s, x_s^v, v_s) dW_s - \int_0^t \frac{\partial \sigma}{\partial t}(t, s, x_s^u, u_s) dW_s \right) dt \right] \\ &= \int_0^T E \left[\int_0^t [D_s p_2(t) | F_s] \left(\frac{\partial \sigma}{\partial t}(t, s, x_s^v, v_s) - \frac{\partial \sigma}{\partial t}(t, s, x_s^u, u_s) \right) E dt \right] ds \\ &= E \left[\int_0^T \left[\int_t^T E[D_t p_2(s) | F_t] \left(\frac{\partial \sigma}{\partial s}(t, s, x_s^v, v_s) - \frac{\partial \sigma}{\partial s}(t, s, x_s^u, u_s) \right) ds \right] dt \right], \end{aligned}$$

and by equality (4), we end up with

$$\begin{aligned} & E \left[\int_0^T p_2(t) \left(\int_0^t \frac{\partial \sigma}{\partial t}(t, s, x_s^v, v_s) dW_s - \int_0^t \frac{\partial \sigma}{\partial t}(t, s, x_s^u, u_s) dW_s \right) dt \right] \\ &= \int_0^T E \left[\int_0^t E[D_s p_2(t) | F_s] \left(\frac{\partial \sigma}{\partial t}(t, s, x_s^v, v_s) - \frac{\partial \sigma}{\partial t}(t, s, x_s^u, u_s) \right) dt \right] ds \\ &= E \left[\int_0^T \int_t^T q_2(s, t) \left(\frac{\partial \sigma}{\partial s}(t, s, x_s^v, v_s) - \frac{\partial \sigma}{\partial s}(t, s, x_s^u, u_s) \right) ds dt \right]. \end{aligned} \tag{36}$$

Substituting (35) and (36) in (34), yields

$$\begin{aligned} & E[p_2(T)(x_T^v - x_T^u)] \\ &= E \left[\int_0^T p_2(t) [b(t, t, x_t^v, v_t) - b(t, t, x_t^u, u_t)] dt \right] \\ &+ E \left[\int_0^T \left[\int_t^T p_2(s) \left(\frac{\partial b}{\partial s}(t, s, x_s^v, v_s) - \frac{\partial b}{\partial s}(t, s, x_s^u, u_s) \right) ds \right] dt \right] \\ &+ E \left[\int_0^T \left[\int_t^T q_2(s, t) \left(\frac{\partial \sigma}{\partial s}(t, s, x_s^v, v_s) - \frac{\partial \sigma}{\partial s}(t, s, x_s^u, u_s) \right) ds \right] dt \right] \\ &- E \left[\int_0^T (x_t^v - x_t^u) \left[\frac{\partial l}{\partial x}(t, x_t^u, u_t) p_1(t) + \frac{\partial b}{\partial x}(t, t, x_t^u, u_t) p_2(t) \right] dt \right] \end{aligned}$$

$$\begin{aligned}
& -E \left[\int_0^T (x_t^v - x_t^u) \left[\int_t^T \left(\frac{\partial^2 b}{\partial s \partial x}(s, t, x_t^u, u_t) \right) p_2(s) ds \right] dt \right] \\
& -E \left[\int_0^T (x_t^v - x_t^u) \left[\frac{\partial \sigma}{\partial x}(t, t, x_t^u, u_t) q_2(t, t) + \int_t^T \frac{\partial^2 \sigma}{\partial s \partial x}(s, t, x_t^u, u_t) q_2(s, t) ds \right] dt \right] \\
& +E \left[\int_0^T [\sigma(t, t, x_t^v, v_t) - \sigma(t, t, x_t^u, u_t)] q_2(t, t) dt \right].
\end{aligned}$$

Then

$$\begin{aligned}
& E[p_2(T)(x_T^v - x_T^u)] \\
& = E \left[\int_0^T p_2(t) [b(t, t, x_t^v, v_t) - b(t, t, x_t^u, u_t)] dt \right] \\
& +E \left[\int_0^T q_2(t, t) [\sigma(t, t, x_t^v, v_t) - \sigma(t, t, x_t^u, u_t)] dt \right] \\
& +E \left[\int_0^T \left[\int_t^T p_2(s) \left(\frac{\partial b}{\partial s}(t, s, x_s^v, v_s) - \frac{\partial b}{\partial s}(t, s, x_s^u, u_s) \right) ds \right] dt \right] \\
& +E \left[\int_0^T \left[\int_t^T q_2(s, t) \left(\frac{\partial \sigma}{\partial s}(t, s, x_s^v, v_s) - \frac{\partial \sigma}{\partial s}(t, s, x_s^u, u_s) \right) ds \right] dt \right] \\
& -E \left[\int_0^T \tilde{H}_x^\theta \left(t, \xi_t^u, x_t^u, u_t, \vec{p}(t), \vec{q}(t, t) \right) (x_t^v - x_t^u) dt \right].
\end{aligned} \tag{37}$$

By replacing (33) and (37) into (32), we get

$$\begin{aligned}
& J^\theta(v) - J^\theta(u) \\
& \geq E \left[\int_0^T \tilde{H}^\theta \left(t, \xi_t^v, x_t^v, v_t, \vec{p}(t), \vec{q}(t, t) \right) dt \right] \\
& -E \left[\int_0^T \tilde{H}^\theta \left(t, \xi_t^u, x_t^u, u_t, \vec{p}(t), \vec{q}(t, t) \right) dt \right] \\
& -E \left[\int_0^T \tilde{H}_x^\theta \left(t, \xi_t^u, x_t^u, u_t, \vec{p}(t), \vec{q}(t, t) \right) (x_t^v - x_t^u) dt \right].
\end{aligned} \tag{38}$$

Since the Hamiltonian $\tilde{H}^\theta$ is convex with respect to (x, v) , we have

$$\begin{aligned}
& E \left[\int_0^T \tilde{H}^\theta \left(t, \xi_t^v, x_t^v, v_t, \vec{p}(t), \vec{q}(t, t) \right) dt \right] \\
& -E \left[\int_0^T \tilde{H}^\theta \left(t, \xi_t^u, x_t^u, u_t, \vec{p}(t), \vec{q}(t, t) \right) dt \right] \\
& \geq E \left[\int_0^T \tilde{H}_x^\theta \left(t, \xi_t^u, x_t^u, u_t, \vec{p}(t), \vec{q}(t, t) \right) (x_t^v - x_t^u) dt \right] \\
& +E \left[\int_0^T \tilde{H}_v^\theta \left(t, \xi_t^u, x_t^u, u_t, \vec{p}(t), \vec{q}(t, t) \right) (v_t - u_t) dt \right],
\end{aligned}$$

or equivalently

$$\begin{aligned}
& E \left[\int_0^T \tilde{H}^\theta \left(t, \xi_t^v, x_t^v, v_t, \vec{p}(t), \vec{q}(t, t) \right) dt \right] \\
& - E \left[\int_0^T \tilde{H}^\theta \left(t, \xi_t^u, x_t^u, u_t, \vec{p}(t), \vec{q}(t, t) \right) dt \right] \\
& - E \left[\tilde{H}_x^\theta (x_t^v - x_t^u) dt \right] \\
& \geq E \left[\int_0^T \tilde{H}_v^\theta \left(t, \xi_t^u, x_t^u, u_t, \vec{p}(t), \vec{q}(t, t) \right) (v_t - u_t) dt \right].
\end{aligned}$$

Then, by using the above inequality in (38), we obtain

$$J^\theta(v) - J^\theta(u) \geq E \left[\int_0^T \tilde{H}_v^\theta \left(t, \xi_t^u, x_t^u, u_t, \vec{p}(t), \vec{q}(t, t) \right) (v_t - u_t) dt \right].$$

In virtue of the necessary optimality conditions (14), the last inequality implies that $J^\theta(v) - J^\theta(u) \geq 0$. Thus the theorem is proved.

Remark 3.2 For the last step of the proof, and according to (35), we have

$$J^\theta(v) - J^\theta(u) \geq E \left[\int_0^T \theta V_t^\theta \tilde{H}_v^\theta \left(t, x_t^u, u_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t) \right) (v_t - u_t) dt \right],$$

we know that $\theta V_t^\theta > 0$, then the above equation can be rewritten as

$$J^\theta(v) - J^\theta(u) \geq E \left[\int_0^T \tilde{H}_v^\theta \left(t, x_t^u, u_t, \tilde{p}_2(t), \tilde{q}_2(t, t), V^\theta(t) \right) (v_t - u_t) dt \right].$$

In virtue of the necessary optimality conditions (30), then the last inequality implies that $J^\theta(v) - J^\theta(u) \geq 0$.

4 Conclusion

This paper contains two main results. The first one, Theorem 3.2, establishes the necessary optimality conditions for the system forward SDVE given in form of risk-sensitive performance criterion, using an almost similar scheme as in Chala [10]. The second main result, Theorem 3.3 and especially the Remark 3.2, which suggests the sufficient optimality conditions of forward SDVE in the form of a risk-sensitive performance criterion. The proof is based on the convexity conditions of the Hamiltonian function, and the initial term of the performance function. It should be noted that the risk-sensitive control problems studied by Lim and Zhou in [12], the results of Hafayed *et al.* [10, 6, 9] are different from ours. Now, in the case where the system is governed by backward Volterra stochastic control system (such result will be discussed in our future work).

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