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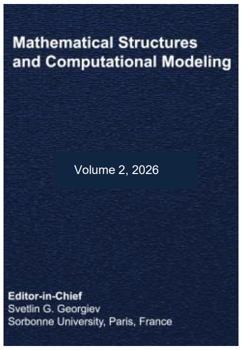

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Optimal Quadratic Refinements of Bernoulli’s Inequality and Some Related Results

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ABSTRACT

Bradley (2025) proved a quadratic strengthening of the classical Bernoulli inequality for integer exponents, and Andrusenko, Shevchuk and Wójcik (2025) established a convex-function equivalence theorem for three functions. This note collects a clean set of consequences and extensions. We prove that Bradley’s quadratic coefficient is optimal on its natural domain, extend the forward implication to k nonnegative functions in a pointwise form, establish the real-exponent quadratic inequality

$$(1+x)^{\alpha} \geq 1+\alpha x+(\alpha-1)x^{2}$$

for every $\alpha \geq 2$ and $x \geq -1$ with best possible coefficient $\alpha-1$, derive quadratic strengthenings and numerical corollaries, show that no positive quadratic coefficient can hold globally for $1<\alpha<2$, and derive the corresponding operator inequalities for self-adjoint operators.

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1. Introduction

The classical Bernoulli inequality states that for every integer $n \geq 1$ and every $x \geq -1$,

$$(1+x)^n \geq 1+nx.$$

Bradley [1] strengthened this by proving that for $n \geq 2$ one has

$$(1+x)^n \geq 1+nx+\lfloor n/2 \rfloor x^2, \tag{1}$$

with the expected domain split: the inequality holds on all of $\mathbb{R}$ when n is even, and on $[-2,\infty)$ when n is odd. Andrusenko, Shevchuk and Wójcik [2] proved a three-function convex-function equivalence theorem of the form

$$f(t)+g(t) \geq 1 \Leftrightarrow f(t)^\alpha + \alpha g(t) \geq 1,$$

under the hypotheses $f(0)=1$, $g(0)=0$, $f, g \geq 0$ and $\alpha > 1$.

The present work records the following observations.

I. Bradley's coefficient $\lfloor n/2 \rfloor$ is optimal on its natural domain.

II. The forward implication of the convex-function equivalence of Andrusenko, Shevchuk and Wójcik [2] extends pointwise to k nonnegative functions; convexity is not needed for that direction.

III. For every real $\alpha \geq 2$ and $x \geq -1$,

$$(1+x)^\alpha \geq 1+\alpha x+(\alpha-1)x^2,$$

and the coefficient $\alpha - 1$ is best possible.

IV. The preceding scalar estimate yields quadratic strengthenings and numerical corollaries.

V. If $1 < \alpha < 2$, no global positive quadratic coefficient exists.

VI. The scalar inequalities lift to self-adjoint operators through the continuous functional calculus.

2. Preliminaries

Throughout, α denotes a real number with $\alpha > 1$ unless stated otherwise. The symbol J denotes a nonempty set; no topological or algebraic structure on J is required unless explicitly stated.

Lemma 2.1 For $\alpha > 1$ define $\varphi: [0,\infty) \to \mathbb{R}$ by

$$\varphi(u) = u^\alpha - \alpha u + (\alpha - 1).$$

Then $\varphi(u) \geq 0$ for all $u \geq 0$, with equality if and only if $u = 1$.

Proof. We have $\varphi'(u) = \alpha(u^{\alpha-1} - 1)$, so φ' is negative on $(0,1)$ and positive on $(1,\infty)$. Hence $u = 1$ is the unique global minimum. Since $\varphi(1) = 0$, the conclusion follows.

Lemma 2.2 For $u_1, \dots, u_k \geq 0$, $v \geq 0$, and $\alpha > 1$,

$$\sum_{i=1}^{k} u_i^\alpha + \alpha v = \sum_{i=1}^{k} \varphi(u_i) + \alpha[\sum_{i=1}^{k} u_i + v] - k(\alpha - 1). \tag{2}$$

Proof. Expand $\varphi(u_i) = u_i^\alpha - \alpha u_i + (\alpha - 1)$, sum over i, and add αv.

3. Main Results

Theorem 3.1 Let $n \geq 2$ be an integer and set $q = \lfloor n/2 \rfloor$. Define

$$D_n = \begin{cases} \mathbb{R}, & if\ n\ is\ even, \\ [-2, \infty), & if\ n\ is\ odd. \end{cases}$$

Then

$$(1+x)^n \geq 1 + nx + qx^2 \qquad for\ all\ x \in D_n,$$

and $q = \lfloor n/2 \rfloor$ is the largest constant for which this inequality holds on D_n.

Proof. The sufficiency is Bradley's theorem [1]; we do not reproduce it here.

For optimality, let $c \in \mathbb{R}$ be such that

$$(1+x)^n \geq 1 + nx + cx^2 \qquad \text{for all } x \in D_n.$$

Since $x = -2 \in D_n$, we may substitute $x = -2$:

$$(-1)^n \geq 1 - 2n + 4c.$$

If n is even, then $1 \geq 1 - 2n + 4c$, so $c \leq n/2 = \lfloor n/2 \rfloor$. If n is odd, then $-1 \geq 1 - 2n + 4c$, so $c \leq (n-1)/2 = \lfloor n/2 \rfloor$. Therefore no constant larger than $\lfloor n/2 \rfloor$ can work.

Theorem 3.2 Let $k \geq 1$, $\alpha > 1$, and let $u_1, \dots, u_k, v: J \to [0, \infty)$ be arbitrary nonnegative functions. If

$$\sum_{i=1}^{k} u_i(t) + v(t) \geq k \qquad \text{for all } t \in J,$$

then

$$\sum_{i=1}^{k} u_i(t)^\alpha + \alpha v(t) \geq k \qquad \text{for all } t \in J.$$

In particular, the implication holds for convex functions, although convexity is not needed for this direction.

Proof. Apply Lemma 2.2 pointwise with $u_i = u_i(t)$ and $v = v(t)$. Since $\varphi(u_i(t)) \geq 0$ by Lemma 2.1, we obtain

$$\sum_{i=1}^{k} u_i(t)^\alpha + \alpha v(t) \geq \alpha[\sum_{i=1}^{k} u_i(t) + v(t)] - k(\alpha - 1) \geq \alpha k - k(\alpha - 1) = k.$$

Remark 3.3 The implication in Theorem 3.2 is genuinely one-way in this paper. It does not follow from the algebraic identity alone. For example, with $k = 1$, $\alpha = 2$, $u_1 = 0.5$ and $v = 0.375$, one has

$$u_1^2 + 2v = 0.25 + 0.75 = 1 \geq 1,$$

but

$$u_1 + v = 0.875 < 1.$$

Thus no converse can be inferred from the pointwise algebraic identity alone.

Theorem 3.4 For every $\alpha \geq 2$ and every $x \geq -1$,

$$(1+x)^{\alpha} \geq 1+\alpha x+(\alpha-1)x^{2}.$$

The coefficient $\alpha-1$ is best possible. If $\alpha=2$, equality holds for all $x \geq -1$. If $\alpha>2$, equality holds if and only if $x \in \{-1,0\}$.

Proof. Set $y=1+x \geq 0$. The inequality is equivalent to

$$y^{\alpha}-(\alpha-1)y^{2}+(\alpha-2)y \geq 0.$$

Define

$$H(y):=y^{\alpha}-(\alpha-1)y^{2}+(\alpha-2)y=yG(y),$$

where

$$G(y):=y^{\alpha-1}-(\alpha-1)y+(\alpha-2).$$

If $\alpha=2$, then $G(y) \equiv 0$ and hence $H(y) \equiv 0$, so equality holds for all $y \geq 0$.

Assume now that $\alpha>2$. Then

$$G'(y)=(\alpha-1)(y^{\alpha-2}-1).$$

Hence $G'(y)<0$ for $0<y<1$ and $G'(y)>0$ for $y>1$. Therefore G attains its global minimum on $[0,\infty)$ at $y=1$. Since

$$G(1)=1-(\alpha-1)+(\alpha-2)=0,$$

we have $G(y) \geq 0$ for all $y \geq 0$. Consequently $H(y)=yG(y) \geq 0$ for all $y \geq 0$.

Equality occurs when either $y=0$ or $G(y)=0$. For $\alpha>2$, the unique zero of G is $y=1$, so equality holds iff $y \in \{0,1\}$, i.e. iff $x \in \{-1,0\}$.

Optimality follows by taking $x=-1$: if $C>\alpha-1$, then

$$0=(1-1)^{\alpha} \geq 1-\alpha+C=C-(\alpha-1)>0,$$

a contradiction.

Remark 3.5 For integer $n \geq 3$, the coefficient $n-1$ in Theorem 3.4 is strictly larger than Bradley's $\lfloor n/2 \rfloor$, but it is valid only on the narrower domain $[-1,\infty)$.

Lemma 3.6 Let $f: J \to [0,\infty)$ be any function and let $\alpha \geq 2$. Then

$$f(t)^{\alpha} \geq 1+\alpha(f(t)-1)+(\alpha-1)(f(t)-1)^{2} \qquad (t \in J).$$

Proof. Apply Theorem 3.4 with $x=f(t)-1 \geq -1$.

Theorem 3.7 Let $\alpha \geq 2$ and let $f,g: J \to [0,\infty)$ satisfy

$$f(t)+g(t) \geq 1 \qquad \text{forall} t \in J.$$

Then

$$f(t)^{\alpha}+\alpha g(t) \geq 1+(\alpha-1)(f(t)-1)^{2} \qquad \text{for all } t \in J.$$

The coefficient $\alpha-1$ is optimal.

Proof. By Lemma 3.6,

$$f(t)^{\alpha} \geq 1 + \alpha(f(t) - 1) + (\alpha - 1)(f(t) - 1)^2.$$

Adding $\alpha g(t)$ gives

$$f(t)^{\alpha} + \alpha g(t) \geq 1 + \alpha(f(t) - 1 + g(t)) + (\alpha - 1)(f(t) - 1)^2.$$

Since $f(t) + g(t) \geq 1$, the middle term is nonnegative, so

$$f(t)^{\alpha} + \alpha g(t) \geq 1 + (\alpha - 1)(f(t) - 1)^2.$$

For optimality, let $J = [0,1]$, $f(t) = 1 - t$ and $g(t) = t$. Then f and g are convex, nonnegative, and satisfy $f(t) + g(t) = 1$ for all $t \in [0,1]$. At $t = 1$ we have

$$f(1)^{\alpha} + \alpha g(1) = 0 + \alpha = \alpha.$$

If a larger coefficient $C > \alpha - 1$ were valid, then at $t = 1$ we would obtain

$$\alpha \geq 1 + C,$$

so $C \leq \alpha - 1$, a contradiction.

Corollary 3.8 Let $k \geq 1$, $\alpha \geq 2$, and let $f_1, \dots, f_k, g: J \to [0, \infty)$ satisfy

$$\sum_{i=1}^{k} f_i(t) + g(t) \geq k \quad \text{for all } t \in J.$$

Then

$$\sum_{i=1}^{k} f_i(t)^{\alpha} + \alpha g(t) \geq k + (\alpha - 1) \sum_{i=1}^{k} (f_i(t) - 1)^2 \qquad \text{for all } t \in J.$$

Proof. Apply Theorem 3.4 to each $f_i(t)$:

$$f_i(t)^{\alpha} \geq 1 + \alpha(f_i(t) - 1) + (\alpha - 1)(f_i(t) - 1)^2.$$

Summing over i and adding $\alpha g(t)$ yields

$$\sum_{i=1}^{k} f_i(t)^{\alpha} + \alpha g(t) \geq k + \alpha[\sum_{i=1}^{k} f_i(t) - k + g(t)] + (\alpha - 1) \sum_{i=1}^{k} (f_i(t) - 1)^2.$$

Since, $\sum f_i(t) + g(t) \geq k$, the bracket is nonnegative. The result follows.

Theorem 3.9 If $1 < \alpha < 2$, there is no constant $C > 0$ such that

$$(1 + x)^{\alpha} \geq 1 + \alpha x + C x^2 \quad \text{for all } x \geq -1.$$

Proof. Assume, to the contrary, that such a constant $C > 0$ exists. Then for every $x > 0$,

$$\frac{(1 + x)^{\alpha} - 1 - \alpha x}{x^2} \geq C.$$

However,

$$\frac{(1 + x)^{\alpha} - 1 - \alpha x}{x^2} = x^{\alpha - 2} \left(1 + \frac{1}{x}\right)^{\alpha} - \frac{1}{x^2} - \frac{\alpha}{x},$$

and since $\alpha - 2 < 0$, the right-hand side tends to 0 as $x \to \infty$. This contradicts the lower bound by the positive constant C. Hence no such C exists.

4. Applications

4.1. Numerical applications

Theorem 4.1 Let $a_1, \ldots, a_k, c \geq 0$ and $\alpha > 1$. If

$$\sum_{i=1}^{k} a_i + c \geq k,$$

then

$$\sum_{i=1}^{k} a_i^{\alpha} + \alpha c \geq k.$$

Proof. Apply Lemma 2.1 to each a_i:

$$a_i^{\alpha} \geq \alpha a_i - (\alpha - 1).$$

Summing over i and adding αc gives

$$\sum_{i=1}^{k} a_i^{\alpha} + \alpha c \geq \alpha[\sum_{i=1}^{k} a_i + c] - k(\alpha - 1) \geq \alpha k - k(\alpha - 1) = k.$$

Remark 4.2 The implication in Theorem 4.1 cannot be reversed in general. For example, with $k = 1$, $\alpha = 2$, $a_1 = 0.5$ and $c = 0.375$ one has

$$a_1^2 + 2c = 1 \geq 1,$$

but

$$a_1 + c = 0.875 < 1.$$

Theorem 4.3 Let $a_1, \ldots, a_k, c \geq 0$ and $\alpha \geq 2$. If

$$\sum_{i=1}^{k} a_i + c \geq k,$$

then

$$\sum_{i=1}^{k} a_i^{\alpha} + \alpha c \geq k + (\alpha - 1) \sum_{i=1}^{k} (a_i - 1)^2.$$

Proof. Apply Corollary 3.8 to the constant functions $f_i(t) \equiv a_i$ and $g(t) \equiv c$.

4.2 Operator Inequalities

This subsection lifts the scalar results to self-adjoint operators via the continuous functional calculus. This is a standard application of spectral theory; the content lies solely in the scalar inequalities that are being lifted.

Throughout this subsection, H denotes a complex Hilbert space, $B(H)$ the algebra of bounded linear operators on H, I the identity operator, and $\sigma(A)$ the spectrum of $A \in B(H)$. For self-adjoint $A, B \in B(H)$ the notation $A \geq B$ means $\langle (A - B)x, x \rangle \geq 0$ for every $x \in H$.

Lemma 4.4 Let $A \in B(H)$ be self-adjoint and let $p \in C(\sigma(A), \mathrm{R})$. If $p(\lambda) \geq 0$ for every $\lambda \in \sigma(A)$, then $p(A) \geq 0$.

Proof. By [3, Theorem~VIII.2.6], the continuous functional calculus $f \mapsto f(A)$ is a $*$-isomorphism from $C(\sigma(A))$ onto $C^*(A) \subseteq B(H)$. Since $p \geq 0$ on $\sigma(A)$, the continuous function $g := \sqrt{p}$ belongs to $C(\sigma(A), R)$ and satisfies $p = g^2$. Therefore,

$$p(A) = (g^2)(A) = g(A)^* g(A) \geq 0,$$

as claimed.

Theorem 4.5 Let $n \geq 2$ be an integer and let $A \in B(H)$ be self-adjoint. Set $q = \lfloor n/2 \rfloor$ and

$$p_n(\lambda) = (1+\lambda)^n - 1 - n\lambda - q\lambda^2.$$

Then:

a) If n is even, then $(I+A)^n \geq I + nA + qA^2$ without any spectral restriction.

b) If n is odd and $A \geq -2I$, then $(I+A)^n \geq I + nA + qA^2$.

Proof. By Theorem 3.1, $p_n(\lambda) \geq 0$ for all $\lambda \in D_n$, where

$$D_n = \begin{cases} \mathbb{R}, & if\ n\ is\ even, \\ [-2, \infty), & if\ n\ is\ odd. \end{cases}$$

If n is even, then $\sigma(A) \subseteq \mathbb{R} = D_n$, so Lemma 4.4 gives $p_n(A) \geq 0$. If n is odd and $A \geq -2I$, then $\sigma(A) \subseteq [-2, \infty) = D_n$ so again Lemma 4.4 gives $p_n(A) \geq 0$. In either case,

$$(I + A_n) - I - nA - qA^2 = p_n(A) \geq 0.$$

Corollary 4.6 Let $\alpha \geq 2$ and let $A \in B(H)$ be self-adjoint with $A \geq -I$. Since $I + A \geq 0$, the operator $(I+A)^\alpha$ is well defined. Then

$$(I+A)^\alpha \ \geq \ I + \alpha A + (\alpha - 1)A^2.$$

The coefficient $\alpha - 1$ is optimal.

Proof. Since $A \geq -I$, one has $\sigma(A) \subseteq [-1, \infty)$ and hence $I + A \geq 0$. Set $r(\lambda) = (1+\lambda)^\alpha - 1 - \alpha\lambda - (\alpha - 1)\lambda^2$. Then $r \in C(\sigma(A), R)$ and, by Theorem 3.4, $r(\lambda) \geq 0$ for every $\lambda \geq -1$. Since $\sigma(A) \subseteq [-1, +\infty)$, Lemma 4.4 gives $r(A) \geq 0$, which is the stated inequality.

Optimality: in dimension one, $A = -I$ gives $(1 + (-1))^\alpha = 0$ and $1 - \alpha + C$, forcing $C \leq \alpha - 1$.

5. Concluding Remarks

The paper presents a concise collection of quadratic refinements of Bernoulli-type inequalities. Its main contribution is to clarify the optimality of the integer-exponent Bradley coefficient $\lfloor n/2 \rfloor$ on its natural domain and, in the real-exponent setting, the optimality of the coefficient $\alpha - 1$ on the narrower domain $[-1, \infty)$. The pointwise k -function implication and the associated quadratic strengthenings follow from the same auxiliary function $\varphi(u) = u^\alpha - \alpha u + (\alpha - 1)$, while the operator inequalities are obtained by standard applications of the continuous functional calculus to the corresponding scalar estimates.

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