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Optimal Quadratic Refinements of Bernoulli's Inequality and Some Related Results

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ABSTRACT

Bradley (2025) proved a quadratic strengthening of the classical Bernoulli inequality for integer exponents, and Andrusenko, Shevchuk and Wójcik (2025) established a convex-function equivalence theorem for three functions. This note collects a clean set of consequences and extensions. We prove that Bradley's quadratic coefficient is optimal on its natural domain, extend the forward implication to k nonnegative functions in a pointwise form, establish the real-exponent quadratic inequality

$$(1+x)^{\alpha} \geq 1 + \alpha x + (\alpha - 1)x^2$$

for every $\alpha \geq 2$ and $x \geq -1$ with best possible coefficient $\alpha - 1$, derive quadratic strengthenings and numerical corollaries, show that no positive quadratic coefficient can hold globally for $1 < \alpha < 2$, and derive the corresponding operator inequalities for self-adjoint operators.

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1. Introduction

The classical Bernoulli inequality states that for every integer $n \geq 1$ and every $x \geq -1$,

$$(1+x)^n \geq 1+nx.$$

Bradley [1] strengthened this by proving that for $n \geq 2$ one has

$$(1+x)^n \geq 1+nx + [n/2]x^2, \tag{1}$$

with the expected domain split: the inequality holds on all of $\mathbb{R}$ when n is even, and on $[-2, \infty)$ when n is odd. Andrusenko, Shevchuk and Wójcik [2] proved a three-function convex-function equivalence theorem of the form

$$f(t) + g(t) \geq 1 \Leftrightarrow f(t)^\alpha + \alpha g(t) \geq 1,$$

under the hypotheses $f(0) = 1$, $g(0) = 0$, $f, g \geq 0$ and $\alpha > 1$.

The present work records the following observations.

- I. Bradley's coefficient $[n/2]$ is optimal on its natural domain.
- II. The forward implication of the convex-function equivalence of Andrusenko, Shevchuk and Wójcik [2] extends pointwise to k nonnegative functions; convexity is not needed for that direction.
- III. For every real $\alpha \geq 2$ and $x \geq -1$,

$$(1+x)^\alpha \geq 1+\alpha x + (\alpha-1)x^2,$$

and the coefficient $\alpha-1$ is best possible.

- IV. The preceding scalar estimate yields quadratic strengthenings and numerical corollaries.
- V. If $1 < \alpha < 2$, no global positive quadratic coefficient exists.
- VI. The scalar inequalities lift to self-adjoint operators through the continuous functional calculus.

2. Preliminaries

Throughout, α denotes a real number with $\alpha > 1$ unless stated otherwise. The symbol J denotes a nonempty set; no topological or algebraic structure on J is required unless explicitly stated.

Lemma 2.1 For $\alpha > 1$ define $\varphi: [0, \infty) \rightarrow \mathbb{R}$ by

$$\varphi(u) = u^\alpha - \alpha u + (\alpha-1).$$

Then $\varphi(u) \geq 0$ for all $u \geq 0$, with equality if and only if $u = 1$.

Proof. We have $\varphi'(u) = \alpha(u^{\alpha-1} - 1)$, so φ' is negative on $(0,1)$ and positive on $(1, \infty)$. Hence $u = 1$ is the unique global minimum. Since $\varphi(1) = 0$, the conclusion follows.

Lemma 2.2 For $u_1, \dots, u_k \geq 0$, $v \geq 0$, and $\alpha > 1$,

$$\sum_{i=1}^k u_i^\alpha + \alpha v = \sum_{i=1}^k \varphi(u_i) + \alpha \left[\sum_{i=1}^k u_i + v \right] - k(\alpha-1). \tag{2}$$

Proof. Expand $\varphi(u_i) = u_i^\alpha - \alpha u_i + (\alpha-1)$, sum over i , and add αv .

3. Main Results

Theorem 3.1 Let $n \geq 2$ be an integer and set $q = [n/2]$. Define

$$D_n = \begin{cases} \mathbb{R}, & \text{if } n \text{ is even,} \\ [-2, \infty), & \text{if } n \text{ is odd.} \end{cases}$$

Then

$$(1+x)^n \geq 1+nx+qx^2 \quad \text{for all } x \in D_n,$$

and $q = \lfloor n/2 \rfloor$ is the largest constant for which this inequality holds on D_n .

Proof. The sufficiency is Bradley's theorem [1]; we do not reproduce it here.

For optimality, let $c \in \mathbb{R}$ be such that

$$(1+x)^n \geq 1+nx+cx^2 \quad \text{for all } x \in D_n.$$

Since $x = -2 \in D_n$, we may substitute $x = -2$:

$$(-1)^n \geq 1-2n+4c.$$

If n is even, then $1 \geq 1-2n+4c$, so $c \leq n/2 = \lfloor n/2 \rfloor$. If n is odd, then $-1 \geq 1-2n+4c$, so $c \leq (n-1)/2 = \lfloor n/2 \rfloor$. Therefore no constant larger than $\lfloor n/2 \rfloor$ can work.

Theorem 3.2 Let $k \geq 1$, $\alpha > 1$, and let $u_1, \dots, u_k, v: J \rightarrow [0, \infty)$ be arbitrary nonnegative functions. If

$$\sum_{i=1}^k u_i(t) + v(t) \geq k \quad \text{for all } t \in J,$$

then

$$\sum_{i=1}^k u_i(t)^\alpha + \alpha v(t) \geq k \quad \text{for all } t \in J.$$

In particular, the implication holds for convex functions, although convexity is not needed for this direction.

Proof. Apply Lemma 2.2 pointwise with $u_i = u_i(t)$ and $v = v(t)$. Since $\varphi(u_i(t)) \geq 0$ by Lemma 2.1, we obtain

$$\sum_{i=1}^k u_i(t)^\alpha + \alpha v(t) \geq \alpha \left[\sum_{i=1}^k u_i(t) + v(t) \right] - k(\alpha - 1) \geq \alpha k - k(\alpha - 1) = k.$$

Remark 3.3 The implication in Theorem 3.2 is genuinely one-way in this paper. It does not follow from the algebraic identity alone. For example, with $k = 1$, $\alpha = 2$, $u_1 = 0.5$ and $v = 0.375$, one has

$$u_1^2 + 2v = 0.25 + 0.75 = 1 \geq 1,$$

but

$$u_1 + v = 0.875 < 1.$$

Thus no converse can be inferred from the pointwise algebraic identity alone.

Theorem 3.4 For every $\alpha \geq 2$ and every $x \geq -1$,

$$(1+x)^\alpha \geq 1+\alpha x + (\alpha-1)x^2.$$

The coefficient $\alpha - 1$ is best possible. If $\alpha = 2$, equality holds for all $x \geq -1$. If $\alpha > 2$, equality holds if and only if $x \in \{-1, 0\}$.

Proof. Set $y = 1 + x \geq 0$. The inequality is equivalent to

$$y^\alpha - (\alpha - 1)y^2 + (\alpha - 2)y \geq 0.$$

Define

$$H(y) := y^\alpha - (\alpha - 1)y^2 + (\alpha - 2)y = yG(y),$$

where

$$G(y) := y^{\alpha-1} - (\alpha - 1)y + (\alpha - 2).$$

If $\alpha = 2$, then $G(y) \equiv 0$ and hence $H(y) \equiv 0$, so equality holds for all $y \geq 0$.

Assume now that $\alpha > 2$. Then

$$G'(y) = (\alpha - 1)(y^{\alpha-2} - 1).$$

Hence $G'(y) < 0$ for $0 < y < 1$ and $G'(y) > 0$ for $y > 1$. Therefore G attains its global minimum on $[0, \infty)$ at $y = 1$. Since

$$G(1) = 1 - (\alpha - 1) + (\alpha - 2) = 0,$$

we have $G(y) \geq 0$ for all $y \geq 0$. Consequently $H(y) = yG(y) \geq 0$ for all $y \geq 0$.

Equality occurs when either $y = 0$ or $G(y) = 0$. For $\alpha > 2$, the unique zero of G is $y = 1$, so equality holds iff $y \in \{0, 1\}$, i.e. iff $x \in \{-1, 0\}$.

Optimality follows by taking $x = -1$: if $C > \alpha - 1$, then

$$0 = (1 - 1)^\alpha \geq 1 - \alpha + C = C - (\alpha - 1) > 0,$$

a contradiction.

Remark 3.5 For integer $n \geq 3$, the coefficient $n - 1$ in Theorem 3.4 is strictly larger than Bradley's $\lfloor n/2 \rfloor$, but it is valid only on the narrower domain $[-1, \infty)$.

Lemma 3.6 Let $f: J \rightarrow [0, \infty)$ be any function and let $\alpha \geq 2$. Then

$$f(t)^\alpha \geq 1 + \alpha(f(t) - 1) + (\alpha - 1)(f(t) - 1)^2 \quad (t \in J).$$

Proof. Apply Theorem 3.4 with $x = f(t) - 1 \geq -1$.

Theorem 3.7 Let $\alpha \geq 2$ and let $f, g: J \rightarrow [0, \infty)$ satisfy

$$f(t) + g(t) \geq 1 \quad \text{for all } t \in J.$$

Then

$$f(t)^\alpha + \alpha g(t) \geq 1 + (\alpha - 1)(f(t) - 1)^2 \quad \text{for all } t \in J.$$

The coefficient $\alpha - 1$ is optimal.

Proof. By Lemma 3.6,

$$f(t)^\alpha \geq 1 + \alpha(f(t) - 1) + (\alpha - 1)(f(t) - 1)^2.$$

Adding $\alpha g(t)$ gives

$$f(t)^\alpha + \alpha g(t) \geq 1 + \alpha(f(t) - 1 + g(t)) + (\alpha - 1)(f(t) - 1)^2.$$

Since $f(t) + g(t) \geq 1$, the middle term is nonnegative, so

$$f(t)^\alpha + \alpha g(t) \geq 1 + (\alpha - 1)(f(t) - 1)^2.$$

For optimality, let $J = [0,1]$, $f(t) = 1 - t$ and $g(t) = t$. Then f and g are convex, nonnegative, and satisfy $f(t) + g(t) = 1$ for all $t \in [0,1]$. At $t = 1$ we have

$$f(1)^\alpha + \alpha g(1) = 0 + \alpha = \alpha.$$

If a larger coefficient $C > \alpha - 1$ were valid, then at $t = 1$ we would obtain

$$\alpha \geq 1 + C,$$

so $C \leq \alpha - 1$, a contradiction.

Corollary 3.8 Let $k \geq 1$, $\alpha \geq 2$, and let $f_1, \dots, f_k, g: J \rightarrow [0, \infty)$ satisfy

$$\sum_{i=1}^k f_i(t) + g(t) \geq k \quad \text{for all } t \in J.$$

Then

$$\sum_{i=1}^k f_i(t)^\alpha + \alpha g(t) \geq k + (\alpha - 1) \sum_{i=1}^k (f_i(t) - 1)^2 \quad \text{for all } t \in J.$$

Proof. Apply Theorem 3.4 to each $f_i(t)$:

$$f_i(t)^\alpha \geq 1 + \alpha(f_i(t) - 1) + (\alpha - 1)(f_i(t) - 1)^2.$$

Summing over i and adding $\alpha g(t)$ yields

$$\sum_{i=1}^k f_i(t)^\alpha + \alpha g(t) \geq k + \alpha \left[\sum_{i=1}^k f_i(t) - k + g(t) \right] + (\alpha - 1) \sum_{i=1}^k (f_i(t) - 1)^2.$$

Since, $\sum f_i(t) + g(t) \geq k$, the bracket is nonnegative. The result follows.

Theorem 3.9 If $1 < \alpha < 2$, there is no constant $C > 0$ such that

$$(1 + x)^\alpha \geq 1 + \alpha x + Cx^2 \quad \text{for all } x \geq -1.$$

Proof. Assume, to the contrary, that such a constant $C > 0$ exists. Then for every $x > 0$,

$$\frac{(1 + x)^\alpha - 1 - \alpha x}{x^2} \geq C.$$

However,

$$\frac{(1 + x)^\alpha - 1 - \alpha x}{x^2} = x^{\alpha-2} \left(1 + \frac{1}{x} \right)^\alpha - \frac{1}{x^2} - \frac{\alpha}{x},$$

and since $\alpha - 2 < 0$, the right-hand side tends to 0 as $x \rightarrow \infty$. This contradicts the lower bound by the positive constant C . Hence no such C exists.

4. Applications

4.1. Numerical applications

Theorem 4.1 Let $a_1, \dots, a_k, c \geq 0$ and $\alpha > 1$. If

$$\sum_{i=1}^k a_i + c \geq k,$$

then

$$\sum_{i=1}^k a_i^\alpha + \alpha c \geq k.$$

Proof. Apply Lemma 2.1 to each a_i :

$$a_i^\alpha \geq \alpha a_i - (\alpha - 1).$$

Summing over i and adding αc gives

$$\sum_{i=1}^k a_i^\alpha + \alpha c \geq \alpha \left[\sum_{i=1}^k a_i + c \right] - k(\alpha - 1) \geq \alpha k - k(\alpha - 1) = k.$$

Remark 4.2 The implication in Theorem 4.1 cannot be reversed in general. For example, with $k = 1$, $\alpha = 2$, $a_1 = 0.5$ and $c = 0.375$ one has

$$a_1^2 + 2c = 1 \geq 1,$$

but

$$a_1 + c = 0.875 < 1.$$

Theorem 4.3 Let $a_1, \dots, a_k, c \geq 0$ and $\alpha \geq 2$. If

$$\sum_{i=1}^k a_i + c \geq k,$$

then

$$\sum_{i=1}^k a_i^\alpha + \alpha c \geq k + (\alpha - 1) \sum_{i=1}^k (a_i - 1)^2.$$

Proof. Apply Corollary 3.8 to the constant functions $f_i(t) \equiv a_i$ and $g(t) \equiv c$.

4.2 Operator Inequalities

This subsection lifts the scalar results to self-adjoint operators via the continuous functional calculus. This is a standard application of spectral theory; the content lies solely in the scalar inequalities that are being lifted.

Throughout this subsection, H denotes a complex Hilbert space, $B(H)$ the algebra of bounded linear operators on H , I the identity operator, and $\sigma(A)$ the spectrum of $A \in B(H)$. For self-adjoint $A, B \in B(H)$ the notation $A \geq B$ means $\langle (A - B)x, x \rangle \geq 0$ for every $x \in H$.

Lemma 4.4 Let $A \in B(H)$ be self-adjoint and let $p \in C(\sigma(A), \mathbb{R})$. If $p(\lambda) \geq 0$ for every $\lambda \in \sigma(A)$, then $p(A) \geq 0$.

Proof. By [3, Theorem~VIII.2.6], the continuous functional calculus $f \mapsto f(A)$ is a $*$ -isomorphism from $C(\sigma(A))$ onto $C^*(A) \subseteq B(H)$. Since $p \geq 0$ on $\sigma(A)$, the continuous function $g := \sqrt{p}$ belongs to $C(\sigma(A), \mathbb{R})$ and satisfies $p = g^2$. Therefore,

$$p(A) = (g^2)(A) = g(A)^*g(A) \geq 0,$$

as claimed.

Theorem 4.5 Let $n \geq 2$ be an integer and let $A \in B(H)$ be self-adjoint. Set $q = \lfloor n/2 \rfloor$ and

$$p_n(\lambda) = (1 + \lambda)^n - 1 - n\lambda - q\lambda^2.$$

Then:

- a) If n is even, then $(I + A)^n \geq I + nA + qA^2$ without any spectral restriction.
- b) If n is odd and $A \geq -2I$, then $(I + A)^n \geq I + nA + qA^2$.

Proof. By Theorem 3.1, $p_n(\lambda) \geq 0$ for all $\lambda \in D_n$, where

$$D_n = \begin{cases} \mathbb{R}, & \text{if } n \text{ is even,} \\ [-2, \infty), & \text{if } n \text{ is odd.} \end{cases}$$

If n is even, then $\sigma(A) \subseteq \mathbb{R} = D_n$, so Lemma 4.4 gives $p_n(A) \geq 0$. If n is odd and $A \geq -2I$, then $\sigma(A) \subseteq [-2, \infty) = D_n$ so again Lemma 4.4 gives $p_n(A) \geq 0$. In either case,

$$(I + A)^n - I - nA - qA^2 = p_n(A) \geq 0.$$

Corollary 4.6 Let $\alpha \geq 2$ and let $A \in B(H)$ be self-adjoint with $A \geq -I$. Since $I + A \geq 0$, the operator $(I + A)^\alpha$ is well defined. Then

$$(I + A)^\alpha \geq I + \alpha A + (\alpha - 1)A^2.$$

The coefficient $\alpha - 1$ is optimal.

Proof. Since $A \geq -I$, one has $\sigma(A) \subseteq [-1, \infty)$ and hence $I + A \geq 0$. Set $r(\lambda) = (1 + \lambda)^\alpha - 1 - \alpha\lambda - (\alpha - 1)\lambda^2$. Then $r \in C(\sigma(A), \mathbb{R})$ and, by Theorem 3.4, $r(\lambda) \geq 0$ for every $\lambda \geq -1$. Since $\sigma(A) \subseteq [-1, +\infty)$, Lemma 4.4 gives $r(A) \geq 0$, which is the stated inequality.

Optimality: in dimension one, $A = -I$ gives $(1 + (-1))^\alpha = 0$ and $1 - \alpha + C$, forcing $C \leq \alpha - 1$.

5. Concluding Remarks

The paper presents a concise collection of quadratic refinements of Bernoulli-type inequalities. Its main contribution is to clarify the optimality of the integer-exponent Bradley coefficient $\lfloor n/2 \rfloor$ on its natural domain and, in the real-exponent setting, the optimality of the coefficient $\alpha - 1$ on the narrower domain $[-1, \infty)$. The pointwise k -function implication and the associated quadratic strengthenings follow from the same auxiliary function $\varphi(u) = u^\alpha - \alpha u + (\alpha - 1)$, while the operator inequalities are obtained by standard applications of the continuous functional calculus to the corresponding scalar estimates.

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