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Solution of the Millennium Problem Related to the Navier-Stokes Equations

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ABSTRACT

This paper contains a solution to the millennium problem related to the Navier-Stokes problem (NSP) in R^3 without boundaries. In the first part of this paper the author's results on hypersingular integrals are presented. By hypersingular integrals we understand the integrals diverging from the classical analysis point of view. We discuss integral equations with such integrals, the Cauchy integrals with densities that are distributions, and the boundary values of analytic functions with distributional density of the Cauchy integrals. Solving integral equations with hypersingular integrals is used in the second part. The second part of the paper is based on the first, and contains an analysis of the Navier-Stokes problem (NSP) in R^3 without boundaries. It is shown that a nontrivial solution to NSP cannot exist for all time $t > 0$. The only global solution is trivial. The NSP paradox is proved. This paradox shows that there cannot exist a solution to the NSP in R^3 without boundaries for all $t > 0$.

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Introduction

This paper provides a solution to the millennium problem concerning the Navier–Stokes equations in $\mathbb{R}^3$.

In Section 1, we develop the theory of hypersingular integrals and solve integral equations with such integrals using the Laplace transform and analytic continuation. Section 2 extends the validity of several classical integral transform formulas via analytic continuation. Section 3 generalizes the Sokhotski–Plemelj formulas to the case where the density is a distribution. Section 4 proves the unique solvability of the Dirichlet problem for the Laplace equation with $L^1(S)$ boundary data using double-layer potentials. Section 5 establishes that the convolution operator $\Phi_\lambda *$ has spectral radius zero and derives a key inequality.

Section 6 analyzes the Navier–Stokes problem in $\mathbb{R}^3$ without boundaries:

$$v_t + (v, \nabla)v = f - \nabla p + \nu \Delta v, \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \quad (1)$$

$$\nabla \cdot v = 0, \quad (2)$$

$$v(x, 0) = v_0(x). \quad (3)$$

Using the theory of hypersingular integrals, we derive an integral inequality for $b(t) = \|\xi \hat{v}(\xi, t)\|$, where the norm is L^2 norm: $\|u\|^2 = \int_{\mathbb{R}^3} |u(x)|^2 dx$, and prove the NSP paradox:

If a smooth, rapidly decaying nontrivial initial condition $v_0(x) \not\equiv 0$ admits a global solution for all $t > 0$, then necessarily $v_0(x) \equiv 0$.

Consequently, the only global solution is the trivial one $v \equiv 0$, which resolves the millennium problem.

In [6] the results are presented independently. References [12]–[15] are added for convenience of the readers.

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1. Hypersingular Integrals, Equations with Such Integrals and Applications

1.1. Hypersingular Integrals

Consider the integral

$$\int_0^t (t-s)^{\lambda-1} b(s) ds = f(t), \quad (4)$$

where $b(s)$ is a smooth function.

For $\lambda > 0$ this integral is defined classically (in classical analysis) as an improper integral. If $\lambda \leq 0$ we call this integral a hypersingular integral. This integral diverges classically. Consider an integral equation with such integral:

$$q(t) = b_0(t) + \int_0^t (t-s)^{\lambda-1} q(s) ds, \quad (5)$$

where $b_0(t)$ is a smooth rapidly decaying function.

Let us define integral (4) for $\lambda \leq 0$. First, assume that $\lambda > 0$. Integral (4) is a convolution $t^{\lambda-1} * b$. Taking the Laplace transform of it one gets $L(t^{\lambda-1} * b) = L(t^{\lambda-1})L(b)$. Let us calculate

$$L(t^{\lambda-1}) := \int_0^\infty e^{-pt} t^{\lambda-1} dt = \frac{1}{p^\lambda} \int_0^\infty e^{-\xi} \xi^{\lambda-1} d\xi = \frac{\Gamma(\lambda)}{p^\lambda}, \quad (6)$$

where $\xi = pt$, $\Gamma(\lambda)$ is the Gamma - function, $Re p > 0$. It is known that $\Gamma(\lambda)$ can be continued analytically from the region $\lambda > 0$ to the complex plane $\mathbb{C}$ and has simple poles at the points $\lambda = 0, -1, -2, \dots$. The function $\frac{1}{\Gamma(\lambda)}$ is an entire function of λ . Therefore $L(t^{-\lambda})L(b) = \frac{L(b)\Gamma(\lambda)}{p^\lambda}$.

This function can be continued analytically in the variable λ on the whole complex plane of λ except for the points $\lambda = 0, -1, -2, \dots$ at which $\Gamma(\lambda)$ has simple poles. Since $L(b)$ does not depend on λ , we define the integral (4) as

$$L^{-1}\left(\frac{\Gamma(\lambda)}{p^\lambda}L(b)\right) = \Gamma(\lambda)L^{-1}\left(\frac{L(b)}{p^\lambda}\right), \quad (7)$$

where L^{-1} is the inverse Laplace transform.

An interesting question related to formula (7) is to determine when the function $L(b(t))$ is the Laplace transform of a function $f(t)$.

Let us state the result which we prove.

Theorem 1.1 Assume that $F(p)$, $p = \sigma + is$, is an analytic function in the half-plane $\sigma > \sigma_0$, that there exists the limit

$$\lim_{\sigma \rightarrow 0} F(\sigma + is)$$

for almost all $s \in \mathbb{R} = (-\infty, \infty)$, that $F(\sigma + is) \in L^1(\mathbb{R})$, and

$$\lim_{|p| \rightarrow \infty} F(p) = 0. \quad (8)$$

uniformly with respect to $|p|$, $|p| \rightarrow \infty, \sigma \geq 0$. Then

$$F(p) = \int_0^\infty e^{-pt} f(t) dt, \quad (9)$$

and

$$f(t) = \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{K_n} e^{pt} F(p) dp, \quad f(t) = 0; \quad t < 0. \quad (10)$$

Here

$$K = K_n = K_{\sigma n} = \{\sigma - in, \sigma + in\}, \quad \sigma \geq \sigma_0 \geq 0,$$

and $|f(t)| \leq C$ for $t \geq 0$.

Proof. Let $L_n = -K_{\sigma n} \cup C_n$, where

$$C_n = b_0 + ne^{i\psi}, \quad -\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2},$$

the contour L_n is oriented counterclockwise. By the analyticity of $F(p)$ and by the Cauchy formula one gets

$$F(p) = \frac{1}{2\pi i} \int_{L_n} \frac{F(q) dq}{q-p}, \quad (11)$$

where p belongs to the bounded domain with the boundary L_n . Let $n \rightarrow \infty$. Then

$$\lim_{n \rightarrow \infty} \left| \int_{C_n} \frac{F(q) dq}{q-p} \right| \leq \frac{\pi n}{n-|p|} \max_{q \in C_n} |F(q)| \rightarrow 0, \quad n \rightarrow \infty, \quad (12)$$

where $G > 0$ is a constant independent of n . We have used assumption (8).

Thus from (11) and (12) one gets

$$F(p) = \frac{1}{2\pi i} \int_{-K} \frac{F(q) dq}{q-p}. \quad (13)$$

One has

$$\frac{1}{q-p} = \int_0^\infty e^{-(q-p)t} dt, \quad \Re q > \Re p. \quad (14)$$

From (13) and (14) one derives

$$F(p) = \int_0^\infty e^{-pt} \left(\frac{1}{2\pi i} \int_K F(q) e^{qt} dq \right) dt = \int_0^\infty e^{-pt} f(t) dt, \quad (15)$$

where

$$f(t) = \frac{1}{2\pi i} \int_K F(q) e^{qt} dq, \quad (16)$$

and

$$F(p) = \int_0^\infty e^{-pt} f(t) dt. \quad (17)$$

Let us prove that $f(t) = 0$ for $t < 0$.

If $h > 0$, then by the analyticity of $F(q)$ and e^{-qt} one has

$$\int_K e^{-qt} F(q) dq = 0. \quad (18)$$

Moreover,

$$\lim_{n \rightarrow \infty} \left| \int_{C_n} e^{-qt} F(q) dq \right| \leq \lim_{n \rightarrow \infty} \left(\sup_{q \in C_n} |F(q)| \right) \int_{-\pi/2}^{\pi/2} e^{-tn \cos \psi} n d\psi = 0, \quad (19)$$

because

$$n \int_{-\pi/2}^{\pi/2} e^{-tn \cos \psi} d\psi = O\left(\frac{1}{t}\right) \rightarrow 0, \quad n \rightarrow \infty, \quad (20)$$

Therefore

$$\frac{1}{2\pi i} \int_K e^{-qt} F(q) dq = 0, \quad t > 0. \quad (21)$$

This and (16) imply $f(t) = 0$ for $t < 0$. If $F(q + is)$ is bounded, then $\sup_{t>0} |f(t)| \leq C$. Theorem 1.1 is proved.

Theorem 1.2 If the assumptions of Theorem 1.1 are satisfied and

$$\lim_{|p| \rightarrow \infty} |F(p)| |p|^\beta = 0, \quad \beta > 1,$$

then $f(t) \in C(\mathbb{R}_+)$ and $f(0) = 0$.

Proof. By formula (17) one has

$$F(p) = \int_0^\varepsilon e^{-pt} f(t) dt + \int_\varepsilon^\infty e^{-pt} f(t) dt = I_1 + I_2, \quad \varepsilon > 0. \quad (22)$$

Moreover,

$$|I_2| \leq C \int_{\varepsilon}^{\infty} e^{-pt} dt = C \frac{e^{-\varepsilon p}}{p}. \quad (23)$$

Let $\varepsilon = |p|^{-1/2}$. Then

$$I_2(p) \leq C \frac{e^{-p^{1/2}}}{p} = O\left(\frac{1}{|p|^{\beta}}\right), \quad |p| \rightarrow \infty,$$

since $\beta > 1$.

This and $F(p) = O\left(\frac{1}{|p|^{\beta}}\right)$, $|p| \rightarrow \infty$, imply

$$I_1(p) = O\left(\frac{1}{|p|^{\beta}}\right), \quad |p| \rightarrow \infty, \quad \beta > 1. \quad (24)$$

One has

$$I_1 = \int_0^{\varepsilon} e^{-pt} f(t) dt = f(0) \frac{1-e^{-p\varepsilon}}{p} + \max_{0 \leq t \leq \varepsilon} |f(t) - f(0)| \frac{1-e^{-p\varepsilon}}{p} = (f(0) + o(1)) \frac{e^{-\varepsilon p}}{p}, \quad |p| \rightarrow \infty. \quad (25)$$

From (24) and (25) it follows that $f(0) = 0$ and $f \in C(\mathbb{R}_+)$. Theorem 1.2 is proved.

Remark 1.1 Assume $\sigma_0 = 0$ and $F(is) \in L^k(\mathbb{R})$, $k > 1$. If $1 \leq k \leq 2$, then (see [5], p. 165)

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(is) e^{-ist} ds \in L^{\frac{k}{k-1}}(\mathbb{R}). \quad (26)$$

By the Hausdorff–Young inequality (see [5], p. 165) one has

$$\|f\|_{L^{\frac{k}{k-1}}} \leq (2\pi)^{\frac{k-1}{k}} \|F\|_{L^k}. \quad (27)$$

If $1 \leq k \leq 2$, then $f(t)$ is a function. If $k > 2$, then f may be a tempered distribution (see [5], pp. 163–164).

Example 1.1 Let

$$F(p) = \frac{1}{(1+p)^{5/4}}.$$

Is this function a Laplace transform? Note that $F(p)$ is analytic in the half-plane $\Re p > 0$ and

$$F(p) = O\left(\frac{1}{|p|^{5/4}}\right), \quad |p| \rightarrow \infty,$$

so $\beta = \frac{5}{4} > 1$ and $F(is) \in L^k(\mathbb{R})$, $1 < k, 2$.

By Theorem 1.1,

$$F(p) = L\{f(t)\}, \quad f(t) = 0 \text{ for } t < 0, \quad f(t) \in C(\mathbb{R}_+).$$

By Theorem 1.2, one concludes that $f(0) = 0$.

Example 1.2 Let

$$F(p) = e^{\sqrt{p}}.$$

We investigate whether $F(p)$ can be represented as the Laplace transform of a function $f(t) \in L^{\ell}(\mathbb{R}_+)$ for some $\ell \geq 1$. Suppose that

$$F(p) = e^{\sqrt{p}} = \int_0^\infty f(t) e^{-pt} dt. \quad (28)$$

Then, by Hölder's inequality with

$$\frac{1}{\ell} + \frac{1}{\ell'} = 1,$$

one has

$$e^{\sqrt{p}} \leq \left(\int_0^\infty |f(t)|^\ell dt \right)^{1/\ell} \left(\int_0^\infty e^{-\ell' p t} dt \right)^{1/\ell'} = C \frac{1}{(\ell' p)^{1/\ell'}}. \quad (29)$$

If $p \rightarrow +\infty$, then the left-hand side grows much faster than the right-hand side of (29). Therefore relation (28) cannot be true for any $f \in L^\ell(\mathbb{R}_+)$, $\ell \geq 1$.

Example 1.3 Consider the integral equation with a hypersingular integral:

$$\varphi(t) = \varphi_0(t) + c \int_0^t (t-s)^{-5/4} \varphi(s) ds, \quad c = \text{const} > 0. \quad (30)$$

Using our definition of hypersingular integrals, the Laplace transform and formula (6), we obtain

$$L\{\varphi\} = \frac{L\{\varphi_0\}}{1 - c\Gamma(-\frac{1}{4})p^{1/4}}.$$

It is known that

$$\Gamma(z+1) = z\Gamma(z).$$

Therefore,

$$\Gamma\left(-\frac{1}{4}\right) = -4, \quad \Gamma\left(\frac{3}{4}\right) =: -c_1, \quad c_1 > 0.$$

Hence,

$$L\{\varphi\} = \frac{L\{\varphi_0\}}{1 + c c_1 p^{1/4}}. \quad (31)$$

If $b_0(t)$ is smooth and rapidly decaying as $t \rightarrow \infty$, then

$$|L(b_0)(p)| = O\left(\frac{1}{|p|}\right), \quad |p| \rightarrow \infty, \quad \Re p > 0.$$

This and formula (31) imply

$$L(\varphi)(p) = O\left(\frac{1}{|p|^{5/4}}\right), \quad |p| \rightarrow \infty.$$

Thus, $\beta = \frac{5}{4} > 1$. By Theorems 1.1 and 1.2 we conclude that $L(\varphi)$ is the Laplace transform of $f(t) \in C(\mathbb{R}_+)$ and $f(0) = 0$.

2. Application of Analytic Continuation to Tables of Integral Transforms

In [1] there are many formulas the validity of which can be greatly extended by analytic continuation. Let us give some examples.

Example 2.1 In [1] formula 1 in Section 2.3 is given:

$$\int_0^\infty x^{-\nu} \sin(yx) dx = y^{\nu-1} \Gamma(1-\nu) \cos\left(\frac{\nu\pi}{2}\right), \quad y > 0, \quad \Re \nu < 2. \quad (32)$$

Classically, the integral in (32) diverges for $\Re v \geq 2$. The right-hand side of (32) admits analytic continuation with respect to v on the complex plane. Since $\Gamma(1-v)$ can be analytically extended to the whole complex plane except at the points

$$v = 1, 2, 3, \dots$$

The function $\cos\left(\frac{v\pi}{2}\right)$ has zeros at the points

$$v = 1 + 2m.$$

These zeros eliminate the poles of $\Gamma(1-v)$ corresponding to odd v .

Example 2.2 Formula 7 in Section 2.4 in [1] is

$$\int_0^\infty x^{-v} e^{-\alpha x} \sin(yx) dx = \Gamma(v) (\alpha^2 + y^2)^{-v/2} \sin\left(\operatorname{varctan} \frac{y}{\alpha}\right), \quad (33)$$

$$\Re \alpha > 0, \quad \Re v > -1.$$

Example 2.3 Formula (4) in Section 2.5 in [1] is

$$\int_0^\infty x^{-v} \ln x \sin(yx) dx = \frac{\pi y^{v-1}}{2\Gamma(1-v)\cos\left(\frac{v\pi}{2}\right)} \left[\frac{\Gamma'(v)}{\Gamma(v)} + \frac{\pi}{2} \cot\left(\frac{v\pi}{2}\right) - \ln y \right], \quad (34)$$

$$|\Re v| < 1.$$

The reader can extend the validity of these formulas by analytic continuation with respect to v of the right-hand sides of these formulas.

3. Singular Integral Equations on a Set of Distributions

Let $D \subset \mathbb{R}^2$ be a bounded connected domain, with S its closed boundary, $S \in C^{1,\alpha}$, $0 < \alpha < 1$, or $S = (-\infty, \infty)$. Consider the singular integral

$$\frac{1}{i\pi} \int_S \frac{\psi(s)}{s-t} ds = B\psi = f. \quad (35)$$

Let $H^m = H^m(S)$ be the Banach space of Hölder-continuous functions on S , $0 < \mu \leq 1$, with the usual norm (see [9]). Let X be the space of bounded functionals on H^m ,

$$\|\psi\|_X = \sup_{\|\varphi\|_{H^m}=1} |(\psi, \varphi)|.$$

It is known [9] that the operator B is bounded on H^m , it has a bounded inverse on H^m , and its range on H^m is the whole space H^m . Therefore, equation (35) is uniquely solvable in H^m .

In this section we generalize these results to the space X . Our basic new assumption is $\psi \in X$.

Let us define

$$\frac{1}{i\pi} \int_S \frac{\psi(s)}{s-z} ds = \left(\psi, \frac{1}{i\pi(s-z)} \right), \quad z \notin S. \quad (36)$$

The parentheses in (36) denote the value of the functional $\psi \in X$ on the smooth element

$$\frac{1}{i\pi(s-z)}$$

defined on S .

We wish to prove an analog of the Sokhotski–Plemelj formulas for $\psi \in X$.

We want to characterize elements $\psi \in X$ which are boundary values of analytic functions

$$\Phi(z) = \frac{1}{2\pi i} \int_S \frac{\psi(s)}{s-z} ds$$

on $\mathbb{C} \setminus S$.

The boundary values of Φ on S we define as follows.

$$(\Phi^\pm, \psi) = \lim_{\delta \rightarrow 0} (\Phi(t \pm \delta N_t), \psi(t)), \quad \forall \psi \in H^m, \quad (37)$$

where $(\Phi^\pm, \psi)$ is the value of $(\Phi^\pm \in X$ on the element $\psi \in H^m$, and N_t is the unit normal to S at the point $t \in S$. The vector N_t is directed into the domain D^+ . Since $\delta > 0$, the point $t + \delta N_t \in D^+$ and $t - \delta N_t \in D^-$. The domain D^- is the complement of D^+ in $\mathbb{R}^2$.

Let us prove the existence of the limits in (37). We have

$$\lim_{\delta \rightarrow 0} \left(\frac{1}{2\pi i} \int_S \frac{\varphi(s) ds}{s-t \mp \delta N_t}, \psi(t) \right) = \left(\varphi, \frac{1}{2\pi i} \int_S \frac{\psi(s) ds}{s-t} \pm \frac{\psi(t)}{2} \right), \quad (38)$$

$$\forall \psi \in H^m,$$

where we have used the known Sokhotski–Plemelj formulas. Therefore, the limits in (38) exist. The right-hand side of (38) is the value of $\varphi \in X$ on an element of H^m , because if $\psi \in H^m$, then

$$\frac{1}{2\pi i} \int_S \frac{\psi(s) ds}{s-t} \in H^m.$$

From formula (38) it follows that

$$(\Phi^+ - \Phi^-, \psi) = (\varphi, \psi), \quad \forall \psi \in H^m, \quad (39)$$

and

$$(\Phi^+ + \Phi^-, \psi) = (\varphi, B\psi), \quad \forall \psi \in H^m. \quad (40)$$

The Sokhotski–Plemelj formulas for $\varphi \in X$ can be written, using formula (38), as follows:

$$\Phi^\pm = \frac{1}{2} B\varphi \pm \frac{\varphi}{2} := \Phi \pm \frac{\varphi}{2}, \quad (41)$$

where $B\varphi$ is defined for $\varphi \in X$ by formula (36) with $\phi = \psi$.

If $\varphi \in L^1(S)$, then $\varphi \in X$. Therefore, formulas (41) are valid for $\varphi \in L^1(S)$.

Let us characterize elements $\varphi \in X$ which are the boundary values of analytic functions in D functions.

If $\varphi = \Phi^+$, then

$$\varphi = \frac{1}{2} B\varphi + \frac{\varphi}{2},$$

that is,

$$B\varphi - \varphi = 0. \quad (42)$$

Theorem 3.1 Formula (42) is a necessary and sufficient condition for $\varphi \in X$ to be a boundary value of an analytic function in D . If (42) holds, then

$$\Phi(z) = 0 \quad \text{in } D^-.$$

4. Dirichlet Problem with $L^1(S)$ Boundary Data

Let D be a connected bounded domain in $\mathbb{R}^3$, S its closed smooth boundary. The problem is

$$\Delta u = 0 \quad \text{in } D, \quad (43)$$

$$u|_S = h \in L^1(S). \quad (44)$$

This problem has been studied by many authors. These basic directions of research concern non-smooth boundaries and non-smooth boundary values. We deal with the non-smooth boundary values. The operator in (43) is a simple elliptic operator, the boundary S is smooth, the boundary values h are non-smooth.

In the literature the boundary values are often smoother than $L^1(S)$ (see for example [7], [8]).

We look for the solution to problem (43)–(44) in the form

$$u(x) = \int_S \frac{\partial g(x,s)}{\partial N_s} \mu(s) ds, \quad (45)$$

where $N = N_s$ is the normal to S at the point $s \in S$, directed out of D , and $\mu = \mu(s)$ is to be found.

The function

$$g(x, y) = -\frac{1}{2\pi} \log|x - y|, \quad x, y \in \mathbb{R}^2,$$

or

$$g(x, y) = \frac{1}{4\pi|x-y|}, \quad x, y \in \mathbb{R}^3,$$

is the fundamental solution. The function defined by (45) solves equation (43).

To shorten the presentation we assume that $x, y \in \mathbb{R}^2$. Then the kernel of the operator

$$A(x, s) = \frac{\partial g(x, s)}{\partial N_s}$$

has a limit as $x \rightarrow t$ along the normal N_t . This limit

$$A(t, s) = \frac{\partial g(t, s)}{\partial N_s}$$

is a continuous function of t and s on S .

Therefore, the integral equation for μ which guarantees that boundary condition (44) holds is

$$-\frac{\mu}{2} + A\mu = h, \quad A\mu = \int_S A(t, s)\mu(s)ds. \quad (46)$$

The integral in (46) is well defined for $\mu \in L^1(S)$ because the kernel $A(t, s)$ is continuous and, therefore, a bounded

function on S .

Let us formulate a criterion of compactness in $L^1(S)$ due to M. Riesz.

Proposition 4.1 A bounded set $M \subset L^1(S)$ is compact in $L^1(S)$ if and only if for an arbitrarily small $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\tau| < \delta$, then for any $h \in M$ one has

$$\|h(s + \tau) - h(s)\|_{L^1(S)} < \varepsilon, \quad s + \tau \in S.$$

Here the norm is the $L^1(S)$ norm:

$$\|h\|_{L^1(S)} = \int_S |h(s)| ds.$$

Lemma 4.1 If $M \subset L^1(S)$, then AM is compact in $L^1(S)$.

Proof.

$$\|(Ah)(t + \tau) - Ah(t)\|_{L^1(S)} \leq |S| \sup_{s, t \in S} |A(t + \tau, s) - A(t, s)| \|h\|_{L^1(S)} < \varepsilon. \quad (47)$$

By Proposition 4.1 the set AM is compact in $L^1(S)$. Lemma 4.1 is proved.

We leave it to the reader to check that $A(t, s)$ is a continuous function of $t, s \in S$ assuming that S is a smooth closed boundary (curve).

If A is compact in $L^1(S)$, then the operator

$$-\frac{1}{2}I + A$$

is of Fredholm type. Let us check that its null-space is trivial, and if this is done, then the operator

$$\left(-\frac{1}{2}I + A\right)^{-1}$$

is bounded in $L^1(S)$.

Lemma 4.2 If $\mu \in L^1(S)$ and

$$\left(-\frac{1}{2}I + A\right)\mu = 0,$$

then $\mu = 0$.

Proof. If $\mu \in L^1(S)$, $t \in \mathbb{R}^2$ and

$$A\mu = \frac{1}{2}\mu, \quad (48)$$

then $\mu \in C(S)$. If $\mu \in C(S)$ and S is smooth, then in [11] it is proved that the limit of (45) as $x \rightarrow t$ along N_t is equal to

$$\left(-\frac{1}{2}I + A\right)\mu.$$

If $\mu \in C(S)$, then it is known (see, for example, [11]) that

$$\left(-\frac{1}{2}I + A\right)\mu = 0 \quad \text{implies} \quad \mu = 0.$$

Lemma 4.2 is proved.

Remark 4.1 The assumption $h \in L^1(S)$ is essential for the uniqueness theorem.

Example 4.1 Let

$$D = \{(x, y): (x - 1)^2 + y^2 < 1\}, \quad S = \{(x, y): (x - 1)^2 + y^2 = 1\},$$

and

$$u = 1 - 2\Re\left(\frac{1}{z}\right).$$

Then

$$u \neq 0 \quad \text{in } D, \quad u|_S = 0 \quad \text{except at the point } x = 0, y = 0,$$

and

$$u|_S \notin L^1(S).$$

In this example,

$$u = 1 - \frac{2x}{x^2 + y^2} = \frac{x^2 + y^2 - 2x}{x^2 + y^2} = 0 \quad \text{on } S,$$

except at the point $x = y = 0$.

Then $h = u|_S$. In polar coordinates $x = r \cos \varphi$, $y = r \sin \varphi$,

$$h = \frac{r^2 - 2r \cos \varphi}{r^2} = \frac{r - 2 \cos \varphi}{r}.$$

Clearly,

$$h \notin L^1(S).$$

Thus, the assumption $h \in L^1(S)$ is a necessary condition for the uniqueness theorem. Uniqueness statement. If $h \in L^1(S)$ and $h = 0$ on S as an element of $L^1(S)$, then $u = 0$ in D .

Let us formulate the result we have proved in this section.

Theorem 4.3 Problem (43)–(44) is uniquely solvable for any $h \in L^1(S)$. The solution exists in the form (45), where $\mu \in L^1(S)$ is the unique solution of equation (46). We assumed so far that $x, y \in \mathbb{R}^n$, $n = 2$. If $n > 2$, then the kernel $A(t, s)$ is not bounded.

The fundamental solution to the equation

$$\Delta g(x, y) = -\delta(x - y), \quad x, y \in \mathbb{R}^n, \quad (49)$$

is

$$g(x, y) = \frac{|x - y|^{2-n}}{(n-2)\omega_n}, \quad n > 2, \quad \omega_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}. \quad (50)$$

Here $\delta(x)$ is the Dirac delta-function, ω_n is the area of the unit sphere in $\mathbb{R}^n$, and $\Gamma(x)$ is the Gamma-function.

Without loss of generality let us assume that $n = 3$. In this case

$$g(x, y) = \frac{1}{4\pi|x-y|}, \quad A(t, s) = \frac{\partial}{\partial v_s} \frac{1}{4\pi|t-s|} = \frac{\cos \varphi}{4\pi|t-s|^2},$$

where φ is the angle between v_s and the vector going from t to s .

Since

$$|A(t, s)| \leq \frac{|\cos \varphi|}{|t-s|^2},$$

the kernel $A(t, s)$ is unbounded when $t \rightarrow s$, because

$$|\cos \varphi| = \left| \cos \left(v_s, \frac{t-s}{|t-s|} \right) \right| = O(|t-s|) \quad \text{when } t \rightarrow s,$$

since we assume that S is smooth.

If $\mu \in L^1(S)$ and $A(t, s)$ is unbounded, the integral $A\mu$ diverges classically.

Definition 4.1

$$(A\mu, \varphi) = (\mu, A^*\varphi), \quad \forall \varphi \in C^\infty(S). \quad (51)$$

where

$$A^*\varphi = \int_S A(t, s) \varphi(t) dt \in C^\infty(S) \quad \text{if } S \in C^\infty.$$

In (51) the generalized function μ acts on a test function $A^*\varphi$ which is smooth.

5. More on Integral Equation (30)

We have solved equation (5) using formula (6). Let us introduce the function

$$\Phi_\lambda(t) = \frac{t^{\lambda-1}}{\Gamma(\lambda)}, \quad \Phi_\lambda(t) = 0 \quad \text{if } t < 0. \quad (52)$$

Formula (6) can be written as

$$L(\Phi_\lambda) = \frac{1}{p^\lambda}, \quad \Re p > 0, \quad \lambda \in \mathbb{R}. \quad (53)$$

Lemma 5.1 If $f \geq 0$ is continuous and $\lambda > 0$, then

$$\Phi_\lambda * f \geq 0. \quad (54)$$

Proof. For $\lambda > 0$ the convolution (54) is defined classically and inequality (54) follows from the assumption $f \geq 0$. Lemma 5.1 is proved.

Theorem 5.2 The operator

$$Af = \Phi_\lambda * f, \quad \lambda > 0,$$

has spectral radius $r = 0$ in the space

$$X_T = C(0, T)$$

for any $T > 0$.

Proof. The spectral radius of a linear operator A is defined by the formula

$$r = \lim_{n \rightarrow \infty} \|A^n\|^{1/n}. \quad (55)$$

We check by induction that

$$\|A^n f\|_{X_T} \leq T^{n\lambda} \frac{1}{\Gamma(n\lambda)} \|f\|_{X_T}, \quad n \geq 1. \quad (56)$$

For $n = 1$ one has

$$\|A f\|_{X_T} \leq \frac{T^\lambda}{\Gamma(\lambda)} \|f\|_{X_T}.$$

Suppose (56) is valid for some $n \geq 1$. Then

$$A^{n+1} f(t) = \int_0^t \frac{(t-s)^{\lambda-1}}{\Gamma(\lambda)} A^n f(s) ds \leq \int_0^t \frac{(t-s)^{\lambda-1}}{\Gamma(\lambda)} \frac{s^{n\lambda} \Gamma(\lambda)}{\Gamma(n\lambda+1)} ds \|f\|_{X_T},$$

hence

$$\|A^{n+1} f\|_{X_T} \leq T^{(n+1)\lambda} \frac{1}{\Gamma((n+1)\lambda)} \|f\|_{X_T}.$$

Thus, the inductive inequality is proved.

Consequently, using the well known asymptotic formula

$$\Gamma(z) = e^{(z-\frac{1}{2}) \ln z - z + \frac{1}{2} \ln(2\pi)} (1 + o(1)), \quad z \rightarrow \infty, \quad (57)$$

one gets

$$r = \lim_{n \rightarrow \infty} \|A^n\|^{1/n} = \lim_{n \rightarrow \infty} \frac{T^\lambda}{[\Gamma(n\lambda)]^{1/n}} = 0. \quad (58)$$

Theorem 5.2 is proved.

Consider the equation

$$q = b_0 + Aq, \quad (59)$$

where A is a linear operator in a Banach space X .

Theorem 5.3 If $r(A) < 1$, then equation (59) is uniquely solvable by iterations for any $b_0 \in X$:

$$q = \sum_{j=0}^{\infty} A^j b_0, \quad (60)$$

and this series converges in X .

Theorem 5.3 is known.

The series (60) converges as a Neumann (geometric) series (see [4]).

Lemma 5.4 Let $b_0 \in C(\mathbb{R}_+)$ and $c_q > 0$. Consider the equation

$$q = b_0 - \frac{1}{c_q} \Phi_{1/4} * q. \quad (61)$$

Then equation (61) has a unique solution

$$q = \sum_{j=0}^{\infty} \left(-\frac{1}{c_q} \Phi_{1/4} * \right)^j b_0, \quad (62)$$

and the series converges at the rate of geometrical series.

Proof. Since the spectral radius of $A := -\frac{1}{c_q} \Phi_{1/4} *$ equals to zero, the series for q converges in the space of continuous functions if b_0 belongs to this space. Lemma 5.4 is proved.

For $f \geq 0$ one has

$$\frac{1}{c_q} \Phi_{1/4} * b_0 \geq 0. \quad (63)$$

Taking into account that spectral radius of the operator $\frac{1}{c_q} \Phi_{1/4} *$ equals to zero by Theorem 5.1, one gets

$$b \leq \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{c_q} \Phi_{1/4} * \right)^n b_0, \quad (64)$$

Consequently,

$$0 \leq b(t) \leq q(t), \quad (65)$$

where q solves equation (61). Recall that $b \geq 0$ and $q > 0$ by definition.

6. Analysis of the Navier–Stokes Problem (NSP) and Solution of a Millennium Problem

6.1. Statement of the NSP

We consider the Navier–Stokes problem

$$\partial_t v + (v \cdot \nabla) v = -\nabla p + \nu \Delta v + f, \text{ in } \mathbb{R}^3 \times \mathbb{R}_+, \quad (66)$$

$$\nabla \cdot v = 0, \text{ in } \mathbb{R}^3 \times \mathbb{R}_+, \quad (67)$$

$$v(x, 0) = v_0(x), \quad \nabla \cdot v_0 = 0, \text{ in } \mathbb{R}^3. \quad (68)$$

Here $v = v(x, t)$, $t \in \mathbb{R}_+ = [0, \infty)$, and $\nu > 0$ is a constant. The functions $v_0(x)$ and $f(x, t)$ are given, while the functions $v = v(x, t)$ and $p = p(x, t)$ are to be determined from equations (66)–(68). The initial data $v_0(x)$ must satisfy the incompressibility condition (67). Our solution is classical, it is a smooth function of x and t .

We aim to derive an integral equation equivalent to the NSP. For this purpose, we introduce the Green's function $G(x, t)$ solving the problem

$$\partial_t G - \nu \Delta_x G = \delta(x) \delta(t) \delta_{jm} - \nabla_x P_m, \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \quad (69)$$

$$\nabla_x \cdot G = 0, \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \quad (70)$$

$$G(x, 0) = 0, \quad \text{in } \mathbb{R}^3. \quad (71)$$

Here $\delta(x)$ and $\delta(t)$ denote Dirac delta functions, and

$$\delta_{jm} = \begin{cases} 1, & j = m, \\ 0, & j \neq m, \end{cases} \quad P_m = P_m(x, t),$$

where P_m is a scalar function. The velocity $v(x, t)$ is a vector-valued function.

We assume

$$P_m(x, t) = 0 \quad \text{for } t \leq 0, \quad \nabla P_m = \frac{\partial P_m}{\partial x_j} e_j$$

Summation over repeated indices is understood.

Before we continue, let us describe a brief history of the Navier–Stokes problem (NSP).

In 1752, Euler proposed equations describing the dynamics of an ideal, non-compressible fluid. Later in 1752, D'Alembert proposed a paradox showing that Euler's equations were physically incorrect. This paradox is discussed in [3].

In 1822, Navier (Claude–Louis Navier) wrote equations (66)–(68) for a viscous incompressible fluid. In 1845, George Gabriel Stokes further studied these equations. The hope was that these equations would allow physicists to avoid contradictions in the theory of fluids.

In 2021, Alexander G. Ramm proved a paradox for the NSP. Ramm's paradox can be formulated as follows.

NSP Paradox: Assume (for simplicity only) that $f = 0$, that $v_0(x) \not\equiv 0$ is a smooth, rapidly decaying function as $|x| \rightarrow \infty$, and that a solution to the NSP exists for all $t \in \mathbb{R}_+$. Then one proves that

$$v(x, t) \equiv 0.$$

Conclusion: This paradox shows that, both physically and mathematically, the NSP in $\mathbb{R}^3$ without boundaries is contradictory and therefore is not satisfactory.

6.2. Derivation of the Integral Equation Equivalent to the NSP

We begin by defining the Fourier transform.

Define the Fourier transform

$$\hat{v}(\xi, t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} v(x, t) e^{-i\xi \cdot x} dx, \quad (72)$$

where $\int := \int_{\mathbb{R}^3}$.

The inverse Fourier transform is

$$v(x, t) = \int_{\mathbb{R}^3} \hat{v}(\xi, t) e^{i\xi \cdot x} d\xi. \quad (73)$$

Let

$$G(x, t) = \int_{\mathbb{R}^3} H(\xi, t) e^{i\xi \cdot x} d\xi. \quad (74)$$

Taking the Fourier transform of equations (69)–(71), we obtain

$$\partial_t H_{jm} + \nu |\xi|^2 H_{jm} = \delta(t) \frac{\delta_{jm}}{(2\pi)^3} - i \xi_j \hat{P}_m(\xi, t), \quad (75)$$

where summation over repeated indices is understood.

Moreover,

$$\xi_j H_{jm} = 0, \quad (76)$$

$$\xi_m \hat{P}_m(\xi, t) = 0, \quad (77)$$

$$H(\xi, 0) = 0. \quad (78)$$

Here

$$\hat{P}_m(\xi, t) = \frac{e_m}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} P(x, t) e^{-i\xi \cdot x} dx. \quad (79)$$

Multiply (75) by ξ_j and use equations (76)–(77), according to which $\xi_j H_{jm} = 0$ and $\xi_j \partial_t H_{jm} = 0$, to obtain

$$0 = \frac{\delta(t)\xi_m}{(2\pi)^3} - i|\xi|^2 \hat{P}_m, \quad (80)$$

or

$$\hat{P}_m = -i \frac{\delta(t)}{(2\pi)^3} \frac{\xi_m}{|\xi|^2}. \quad (81)$$

Therefore,

$$\partial_t H_{jm} + \nu |\xi|^2 H_{jm} = \frac{\delta(t)}{(2\pi)^3} \left(\delta_{jm} - \frac{\xi_j \xi_m}{|\xi|^2} \right). \quad (82)$$

Taking the Laplace transform of (82) and using (77), one finds

$$(p + \nu |\xi|^2) L(H_{jm}) = \frac{1}{(2\pi)^3} \left(\delta_{jm} - \frac{\xi_j \xi_m}{|\xi|^2} \right). \quad (83)$$

Therefore,

$$H_{jm}(\xi, t) = \frac{1}{(2\pi)^3} \left(\delta_{jm} - \frac{\xi_j \xi_m}{|\xi|^2} \right) e^{-\nu |\xi|^2 t}. \quad (84)$$

Consequently, by (74),

$$G_{jm}(x, t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\xi \cdot x} e^{-\nu |\xi|^2 t} \left(\delta_{jm} - \frac{\xi_j \xi_m}{|\xi|^2} \right) d\xi. \quad (85)$$

This function $G(x, t)$ solves equations (69)–(71).

Lemma 6.1 *If*

$$\partial_t w - \nu \Delta w = 0 \text{ in } \mathbb{R}^3 \times \mathbb{R}_+, \quad (86)$$

$$w(x, 0) = w_0(x), \quad (87)$$

then

$$w(x, t) = \int_{\mathbb{R}^3} g(x - y, t) w_0(y) dy. \quad (88)$$

Here

$$g(x, t) = \frac{e^{-\frac{|x|^2}{4\nu t}}}{(4\pi\nu t)^{3/2}}, \quad t > 0, \quad (89)$$

and

$$g(x, 0) := \lim_{t \rightarrow 0^+} g(x, t) = 0. \quad (90)$$

Proof. Taking the Fourier transform of equations (86)–(87), one obtains

$$\partial_t \hat{w} + \nu |\xi|^2 \hat{w} = 0, \quad (91)$$

$$\hat{w}(\xi, 0) = \hat{w}_0(\xi). \quad (92)$$

Solving for $\hat{w}$ yields

$$\hat{w}(\xi, t) = e^{-\nu|\xi|^2 t} \hat{w}_0(\xi). \quad (93)$$

Using the convolution formula

$$F\left[\int_{\mathbb{R}^3} h(x-y) f(y) dy\right] = (2\pi)^3 \hat{h} \hat{f}, \quad (94)$$

and the identity

$$e^{-\nu|\xi|^2 t} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-i\xi \cdot x} g(x, t) dx, \quad (95)$$

we obtain (88), namely

$$w(x, t) = \int_{\mathbb{R}^3} g(x-y, t) w_0(y) dy. \quad (96)$$

Lemma 6.1 is proved.

We assume (for simplicity of writing only) that $f = 0$. Let G be defined by (85) and define

$$v(x, t) = \int_0^t \int_{\mathbb{R}^3} G(x-y, t-s) [f(y, s) - (v \cdot \nabla)v(y, s)] dy ds + \int_{\mathbb{R}^3} g(x-y, t) v_0(y) dy. \quad (97)$$

Theorem 6.2 Equation (97) (with $f = 0$) defines a solution to (66)–(68) with $f = 0$.

Proof. Apply the operator $\partial_t - \nu \Delta$ to (97) and use (69) to obtain

$$\partial_t v - \nu \Delta v = \int_0^t \int_{\mathbb{R}^3} [\delta(x-y) \delta(t-s) \delta_{jm} - \nabla_x P_m(x-y, t-s)] [-(v \cdot \nabla)v]_m dy ds = -(v \cdot \nabla)v(x, t) - \nabla p(x, t), \quad (98)$$

where

$$p(x, t) := - \int_0^t \int_{\mathbb{R}^3} P_m(x-y, t-s) [(v \cdot \nabla)v]_m(y, s) dy ds. \quad (99)$$

By (70) and (68), one has

$$\nabla \cdot v = \nabla \cdot \int_{\mathbb{R}^3} g(x-y, t) v_0(y) dy = \nabla \cdot \int_{\mathbb{R}^3} g(z, t) v_0(x+z) dz = 0, \quad (100)$$

where we set $z = y - x$ and used the symmetry $g(x, t) = g(-x, t)$.

Thus, we have verified that a solution to (97) satisfies equations (66)–(68) with $f = 0$. Theorem 6.2 is proved.

We will work with the Fourier transform of equation (97) with $f = 0$:

$$\hat{v}(\xi, t) = - \int_0^t H(\xi, t-s) \hat{v} * (i\xi \hat{v})(\xi, s) ds + (2\pi)^3 \hat{g}(\xi, t) \hat{v}_0(\xi), \quad (101)$$

where $*$ denotes convolution in $\mathbb{R}^3$.

We compute $\hat{g}(\xi, t)$:

$$\begin{aligned} \hat{g}(\xi, t) &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-i\xi \cdot x} \frac{e^{-\frac{|x|^2}{4\nu t}}}{(4\pi\nu t)^{3/2}} dx \\ &= \frac{1}{(2\pi)^3 (4\pi\nu t)^{3/2}} \prod_{j=1}^3 \int_{-\infty}^{\infty} e^{-i\xi_j x_j} e^{-\frac{x_j^2}{4\nu t}} dx_j \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3 (4\pi\nu t)^{3/2}} \left(2 \int_0^\infty \cos(\xi_j x_j) e^{-\frac{x_j^2}{4\nu t}} dx_j \right)^3 \\
&= \frac{1}{(2\pi)^3 (4\pi\nu t)^{3/2}} \left(\sqrt{4\pi\nu t} e^{-\nu t \xi_j^2} \right)^3 \\
&= \frac{1}{(4\pi)^3} e^{-\nu t |\xi|^2}.
\end{aligned} \tag{102}$$

Here we have used a standard Gaussian integral formula (see, for example, Section 2.4 in [1]).

Consequently,

$$(2\pi)^3 \hat{g}(\xi, t) \hat{v}_0(\xi) = \frac{1}{8} e^{-\nu t |\xi|^2} \hat{v}_0(\xi). \tag{103}$$

Note that by Young's convolution inequality

$$\|\hat{v} * (i\xi \hat{v})\|_1 \leq \|\hat{v}\| \|\xi \hat{v}\|, \tag{104}$$

where the norm is $L^2(\mathbb{R}^3)$ norm.

Denote

$$\|\xi \hat{v}(\xi, t)\| := b(t), \tag{105}$$

and assume that

$$\|v_0\| \leq C. \tag{106}$$

Lemma 6.3 *One has*

$$\sup_{t>0} \|\hat{v}(\xi, t)\| \leq C. \tag{107}$$

Proof. Multiply equation (66) by v and integrate over $\mathbb{R}^3$ to obtain

$$\frac{1}{2} \frac{d}{dt} \|v\|^2 + \int_{\mathbb{R}^3} v_m v_j \partial_j v_m dx = \int_{\mathbb{R}^3} f \cdot v dx - \int_{\mathbb{R}^3} (\nabla p) \cdot v dx + \nu \int_{\mathbb{R}^3} (\Delta v) \cdot v dx. \tag{108}$$

Using equation (67), one gets

$$- \int_{\mathbb{R}^3} (\nabla p) \cdot v dx = \int_{\mathbb{R}^3} p(\nabla \cdot v) dx = 0. \tag{109}$$

Further more,

$$\int_{\mathbb{R}^3} v_m v_j \partial_j v_m dx = \int_{\mathbb{R}^3} v_j \partial_j \left(\frac{v_m^2}{2} \right) dx = - \int_{\mathbb{R}^3} (\partial_j v_j) \frac{v_m^2}{2} dx = 0. \tag{110}$$

Moreover,

$$\nu \int_{\mathbb{R}^3} (\Delta v) \cdot v dx = -\nu \int_{\mathbb{R}^3} |\nabla v|^2 dx < 0. \tag{111}$$

From equations (108)–(111) one obtains

$$\frac{d}{dt} \|v\|^2 \leq 2\|f\| \|v\|. \tag{112}$$

Therefore, $\frac{d}{dt} \|v\| \leq \|f\|$. Integrating with respect to time yields

$$\|v(t)\| \leq \|v_0\| + \int_0^t \|f(s)\| ds. \quad (113)$$

Consequently,

$$\sup_{t>0} \|v(t)\| \leq C, \quad (114)$$

where $C := \|v_0\| + C_1$ and we assume

$$\int_0^\infty \|f(s)\| ds \leq C_1. \quad (115)$$

Therefore estimate (107) and Lemma 6.3 are proved.

Later we will prove the following result.

Theorem 6.4 A solution to (66)–(68) satisfies the following a priori estimate:

$$\sup_{t>0} (\|v(t)\| + \|\nabla v(t)\|) \leq C. \quad (116)$$

Let us derive the integral equation for $b(t)$ defined in (105). First, we need two formulas:

$$\|\xi e^{-vt|\xi|^2}\| \leq \frac{C}{t^{5/4}}, \quad (117)$$

$$\|e^{-vt|\xi|^2}\| = \frac{C'}{t^{3/4}}. \quad (118)$$

Proof of (117)–(118). The proof is straightforward. Indeed,

$$\|\xi e^{-vt|\xi|^2}\|^2 = \int_{\mathbb{R}^3} |\xi|^2 e^{-2vt|\xi|^2} d\xi = 4\pi \int_0^\infty r^4 e^{-2vtr^2} dr = \frac{4\pi}{(2vt)^{5/2}} \int_0^\infty s^4 e^{-s^2} ds = \frac{C}{t^{5/2}}, \quad (119)$$

where $s = (2vt)^{1/2}r$. This proves (117).

Similarly,

$$\|e^{-vt|\xi|^2}\|^2 = \int_{\mathbb{R}^3} e^{-2vt|\xi|^2} d\xi = \frac{C'}{t^{3/2}}, \quad (120)$$

which implies (118).

Let us estimate $\|\hat{v}\|$ using equation (101):

$$\|\hat{v}(t)\| \leq \int_0^t \|H(t-s)\| \|\hat{v}(s)\| \|\xi \hat{v}(s)\| ds + C e^{-vt|\xi|^2}. \quad (121)$$

Here we used the estimate $\sup_{\xi} \|\hat{v}_0(\xi)\| \leq C$. Using formula (91), we have

$$|H(\xi, t)| \leq C e^{-vt|\xi|^2}, \quad (122)$$

where we have used the estimate

$$\left| \delta_{jm} - \frac{\xi_j \xi_m}{|\xi|^2} \right| \leq 1, \quad j, m = 1, 2, 3. \quad (123)$$

Thus, using estimates (107) and (122), we obtain

$$|\hat{v}(\xi, t)| \leq C \int_0^t e^{-v|\xi|^2(t-s)} b(s) ds + C e^{-vt|\xi|^2}, \quad (124)$$

where we assume that $\hat{v}_0(\xi)$ is smooth and rapidly decaying. Multiply (124) by $|\xi|$ and take the $L^2(\mathbb{R}^3)$ norm to obtain

$$b(t) \leq C \int_0^t \frac{b(s)}{(t-s)^{5/4}} ds + b_0(t), \quad (125)$$

where

$$b_0(t) := C \left\| |\xi| e^{-vt|\xi|^2} \hat{v}_0(\xi) \right\|. \quad (126)$$

If one assumes that $|\hat{v}_0(\xi)|$ decays sufficiently fast as $|\xi| \rightarrow \infty$, then

$$0 \leq b_0(t) \leq \frac{C}{(1+t)^m}. \quad (127)$$

Together with inequality (125), we consider the hypersingular integral equation

$$b(t) = C \int_0^t \frac{b(s)}{(t-s)^{5/4}} ds + b_0(t). \quad (128)$$

Equation (128) is solved using the formula derived in Section 1. The hypersingular integral in (128) is defined in Sections 1.1 and 1.2. Taking the Laplace transform of equation (128), one gets $L(b) = C\Gamma\left(-\frac{1}{4}\right)L(b) + L(b_0)$, that is, $L(b) = L(b_0) \left(1 - C\Gamma\left(-\frac{1}{4}\right)\right)^{-1}$. We have used the formula $L(\int_0^t s^{\lambda-1} b(s) ds) = \Gamma(\lambda) p^{-\lambda} L(b)$. Since $\Gamma(s+1) = s\Gamma(s)$, we take $s = -\frac{1}{4}$ and get $\Gamma\left(-\frac{1}{4}\right) = -4\Gamma\left(\frac{3}{4}\right) := -c_1 < 0$. So, $L(b) = L(b_0)(1 + Cc_1 p^{\frac{1}{4}})^{-1}$. If one assumes that $b_0(t)$ is smooth and rapidly decaying, then $L(b(t))$ is smooth and its Laplace transform satisfies

$$\lim_{|p| \rightarrow \infty} (|p|^\beta L(b)(p)) = 0, \quad \beta > 1. \text{ Rep} > 0, \beta = \frac{5}{4}. \quad (129)$$

By Theorem 1.2 one gets

$$b \in C(\mathbb{R}_+), \quad b(0) = 0.$$

This also follows from inequality (125).

Using Lemma 5.4, consider equation (61). Since $t > 0$, one may use equation (60) for b . Inequality (65) allows one to prove the NSP paradox.

Theorem 6.5 (NSP paradox) Assume that $v_0(x) \not\equiv 0$ is smooth and rapidly decaying as $|x| \rightarrow \infty$, and that the solution to the NSP (66)–(68) exists for all $t \in \mathbb{R}_+$. Then

$$v_0(x) \equiv 0.$$

Proof. From (129) and (65) it follows that

$$b(0) = 0. \quad (130)$$

That is,

$$\| |\xi| \hat{v}(\xi, 0) \| = 0. \quad (131)$$

Therefore,

$$|\xi| \hat{v}(\xi, 0) = 0. \quad (132)$$

Since $v \in L^2(\mathbb{R}^3)$, one concludes that $\hat{v}(\xi, 0) \in L^2(\mathbb{R}^3)$ and hence

$$\hat{v}(\xi, 0) = 0.$$

Consequently,

$$v(x, 0) = v_0(x) \equiv 0.$$

Theorem 6.5 is proved.

The NSP paradox proves that the Navier-Stokes equations are physically incorrect and should be replaced by equations free from paradoxes.

Finally, let us prove Theorem 6.4.

Proof. Since we have proved estimate (107), it is sufficient to show that

$$\sup_{t>0} \|\nabla v(x, t)\| \leq C. \quad (133)$$

By Parseval's equality, inequality (133) is equivalent to

$$\sup_{t>0} b(t) \leq C. \quad (134)$$

Inequality (134) follows from estimate (125) and inequality (65). This completes the proof of Theorem 6.4.

6.3. Solution to the Millennium Problem Related to the NSP

This problem can be formulated as follows:

Assume (only for simplicity of writing) that $f = 0$, $v_0(x) \not\equiv 0$ is smooth and rapidly decaying, and that the solution to (66)–(68) exists for all $t > 0$. Can such a solution exist?

If such a solution were to exist, then by Theorem 6.5 one would have

$$v_0(x) \equiv 0.$$

This contradiction (which the author calls the NSP paradox [10]) shows that a nontrivial solution to (66)–(68) may exist for all $t > 0$ if and only if $v_0(x) \equiv 0$. In this case, the solution is trivial.

$$v(x, t) \equiv 0.$$

Our argument does not prevent the existence of a solution corresponding to $v_0(x) \neq 0$ on a finite time interval $[0, T]$, with $T < \infty$.

Our argument is valid for the problem (66)–(68), which is the Navier–Stokes problem in $\mathbb{R}^3$ without boundaries.

7. Conclusion

We prove that the classical solution to the Navier-Stokes problem in the whole $\mathbb{R}^3$ space with the initial data a smooth rapidly decaying function not identically equal to zero cannot exist for all $t > 0$. This provides a solution to the millennium problem related to the Navier-Stokes equations.

Our Theorem 6.5 (the NSP paradox) shows that the Navier-Stokes system is physically and mathematically contradictive. Finding its substitution free from contradictive conclusions is necessary.

Conflict of Interest Statement

There is no conflict of interest.

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