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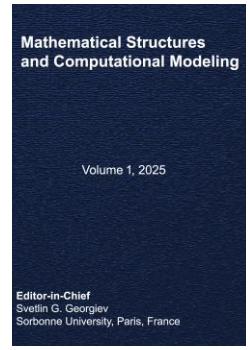

Upper Bound of the Blowing-up Time of κth-order Solution for Nonlinear Wave Equation with Averaged Damping in R^n

Derradji Guidad[1] and Keltoum Bouhali[2,*]

[1]*Department of Mathematics, College of Sciences, University Mohamed Khider - Biskra, Algeria*
[2]*Department of Mathematics, College of Science, Qassim University, Saudi Arabia*

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ABSTRACT

The aim of the present article is to study the nonlinear wave equation with averaged damping in high-order spaces. The concavity method was developed to prove the upper bound of the blowing-up time. To compensate for the lack of the classic Poincaré's inequality in an unbounded domain, we proposed the density function to construct and defined a weighted space.

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*Corresponding Author
Email: k.bouhali@qu.edu.sa;
Tel: +(966) 559355871

1. Introduction and Position of Problem

Let $x \in \boldsymbol{R}^n, n > 3\kappa$ be a space-independent variable and let $t \in [0, +\infty)$ be the times, and for simplicity, we denote the unknowns $u(x,t) = u,\ u^{'}(x,t) = u^{'},\ u^{'} = \frac{du}{dt}$ and $u'' = \frac{d^2u}{dt^2}$ when there is no confusion. The exponent κ will be specified later. In this article we consider the initial value problem

$$u'' + (-1)^{\kappa}\, \emptyset(x)\, \Delta^{\kappa}\, u + v\, \|u'\|^{q-2}_{L^{q-2}_{\rho}(\mathbf{R}^n)}\, u' = |u|^{p-2}u \tag{1.1}$$

where $\kappa \geq 1,\ q, q > 2, v \geq 0$. Equation (1.1) is an important physical model, especially when it comes with nonlinear averaged damping. It is a prototype for a non-linear partial differential equation of hyperbolic type in higher-order spaces equipped with the next initial conditions.

$$u(x,0) = u_0(x) \in \mathrm{D}^{\kappa,2}(\mathbf{R}^n), \quad u'(x,0) = u_1(x) \in L^2_{\rho}(\mathbf{R}^n). \tag{1.2}$$

Here, we assume that $\varphi \in C(\boldsymbol{R}, \boldsymbol{R})$ satisfies

$$(\varphi(x))^{-1} = \rho(x),\ \ \varphi(x) > 0. \tag{1.3}$$

The weighted spaces $D^{\kappa,2}$ and L^2_{ρ} are introduced in Definition 2.2 and the density function $\rho \in C(\boldsymbol{R}^n, \boldsymbol{R})$ satisfies

$$\rho(x) > 0, \quad \rho(x) \in C^{0,\tilde{\gamma}}(\mathbf{R}^n),\ \ n \geq 2\kappa, \tag{1.4}$$

where $0 \leq \tilde{\gamma} \leq 1$ and $\rho \in L^s(\mathbf{R}^n) \cap L^{\infty}(\mathbf{R}^n)$ with $s = \frac{n}{\kappa}$. We mention that

$$|\nabla^{\kappa} u|^2 = (\Delta^{\kappa/2} u)^2,\ \ for\ par\ value\ of\ \kappa,$$

and

$$|\nabla^{\kappa} u|^2 = |\nabla(\Delta^{(\kappa-1)/2} u)|^2,\ \ for\ odd\ \kappa,$$

where

$$|\nabla u|^2 = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2,\ \ \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

For $\kappa = 2$, in an open bounded domain, the authors in [16] proposed

$$u'' +\ \Delta^2 u + \|u'\|^l\, u' = |u|^{p-2}u \tag{1.5}$$

Equation (1.5) is well studied, where the global existence in time is proved and also the blow-up in both negative and positive initial energy is obtained. Recent results for problems with localized nonlinear damping are proposed in [1, 10, 3].

In $\boldsymbol{R}^n$, the article [8] considered a wave equation with frictional damping and a nonlinear source in Kirchhoff type as

$$u'' +\ \emptyset(x)\, \|\nabla u\|^2\, \Delta u + \delta u' = |u|^a u \tag{1.6}$$

with non-positive initial energy, the authors proved that in finite time (well defined) the solution blows up. The results are obtained in weighted spaces. The function spaces with density and their properties are defined and used in [5, 13, 12]. The operator $\Delta^{\kappa},\ \kappa > 1$ and its properties are found in [9] and used in [14] for a coupled system in an open bounded domain Ω. The blow-up phenomena are well studied in [7, 4, 6, 11]. Our article is structured as follows. In Section 2, we introduce some useful results related to the weighted spaces, some useful tools, and state the local existence. Our main result regarding the blow-up of the solution is stated and proved in section 3.

2. Preliminaries and Basic Knowledge

We introduce certain results for the weighted spaces and different embeddings in the high-order spaces.

Definition 2.1 We say that the functions

$$u \in C([0,T]; D^{\kappa,2}(\boldsymbol{R}^n)) \cap C^1([0,T]; L^2_\rho(\boldsymbol{R}^n)),$$

with initial data given in (1.2) are a distributed solution to (1.1) on $[0,T]$, if

$$\begin{aligned}\int_{\boldsymbol{R}^n} \rho u' \psi(x)dx + \int_0^t \left[\int_{\boldsymbol{R}^n} \nabla^\kappa u(x,s) \nabla^\kappa \psi(x)dx + \nu ||u'(s)||^{q-2} \int_{\boldsymbol{R}^n} u'(x,s)\psi(x)dx\, \rho\right] ds \\ = \int_{\boldsymbol{R}^n} \rho u_1 \psi(x)dx + \int_0^t \int_{\boldsymbol{R}^n} \rho |u|^{p-2} u(x,s)\psi(x)dxds,\end{aligned} \tag{2.1}$$

holds for every test function $\psi \in D^{\kappa,2}(\boldsymbol{R}^n)$, $\forall t \in [0,T]$.

Definition 2.2 [13] The function spaces of our problem and its norm are defined as follows:

$$D^{\kappa,2}(\boldsymbol{R}^n) = \{w \in L^{2n/(n-2\kappa)}(\boldsymbol{R}^n) : \nabla^\kappa w \in L^2(\boldsymbol{R}^n),\ n > 3\kappa\}, \tag{2.2}$$

and the space $L^2_\rho(\boldsymbol{R}^n)$ is to be the closure of $C_0^\infty(\boldsymbol{R}^n)$ functions with respect to the inner product

$$(w,v)_{L^2_\rho(\boldsymbol{R}^n)} = \int_{\boldsymbol{R}^n} \rho w v dx.$$

For $q \in (1,\infty)$, if w is a measurable function on $\boldsymbol{R}^n$, we define

$$\|w\|_{L^q_\rho(\boldsymbol{R}^n)} = \left(\int_{\boldsymbol{R}^n} \rho |w|^q dx\right)^{1/q}, \tag{2.3}$$

and

$$\|w\|_{L^q(\boldsymbol{R}^n)} = \left(\int_{\boldsymbol{R}^n} |w|^q dx\right)^{1/q}. \tag{2.4}$$

Then $D^{\kappa,2}(\boldsymbol{R}^n)$ can be embedded continuously in $L^{2n/(n-2\kappa)}(\boldsymbol{R}^n)$, i.e. $\exists k > 0$ where

$$\|w\|_{L^{2n/(n-2\kappa)}(\boldsymbol{R}^n)} \leq k \|w\|_{D^{\kappa,2}(\boldsymbol{R}^n)}. \tag{2.5}$$

The generalized Poincaré's inequality will be used

$$\int_{\boldsymbol{R}^n} |\nabla^\kappa w|^2 dx \geq \gamma \int_{\boldsymbol{R}^n} \rho w^2 dx,\ w \in C_0^\infty(\boldsymbol{R}^n),\ \rho \in L^{n/\kappa}(\boldsymbol{R}^n), \tag{2.6}$$

where $\gamma = k^{-2} \|\rho\|^{-1}_{L^{n/\kappa}(\boldsymbol{R}^n)}$.

The Hilbert space $L^2_\rho(\boldsymbol{R}^n)$ is separable and

$$(w,w)_{L^2_\rho(\boldsymbol{R}^n)} = \|w\|^2_{L^2_\rho(\boldsymbol{R}^n)},$$

consist of all w where

$$\|w\|_{L^q_\rho(\boldsymbol{R}^n)} < \infty,\ 1 < q < +\infty.$$

Let

$$\rho \in L^{n/\kappa}(\boldsymbol{R}^n) \cap L^\infty(\boldsymbol{R}^n),$$

then the embedding $D^{\kappa,2}(\boldsymbol{R}^n) \subset L^2_\rho(\boldsymbol{R}^n)$ is compact and we have

$$\|w\|_{L^2_\rho(\boldsymbol{R}^n)} \le \|\rho\|_{L^{n/\kappa}(\boldsymbol{R}^n)} \|\nabla^\kappa w\|_{L^2(\boldsymbol{R}^n)},\ \forall w \in D^{2,2}(\boldsymbol{R}^n), \tag{2.7}$$

where $\|\rho\|_{L^{n/\kappa}(\boldsymbol{R}^n)} = c_* > 0$.

Lemma 2.3 [[13], Lemma 3.1] Let ρ satisfy (1.4), then $\forall w \in D^{\kappa,2}(\boldsymbol{R}^n)$

$$\|w\|_{L^q_\rho(\boldsymbol{R}^n)} \le \|\rho\|_{L^s(\boldsymbol{R}^n)} \|\nabla^\kappa w\|_{L^2(\boldsymbol{R}^n)}, \tag{2.8}$$

where

$$s = \frac{2n}{2n - qn + q\kappa},$$

for all

$$\begin{cases} 2 \le q < +\infty & if n = \kappa, 2\kappa \\ 2 \le q \le \dfrac{2n}{n - 2\kappa} & if, n \ge 3\kappa. \end{cases} \tag{2.9}$$

For the operator $(-1)^\kappa \varphi \Delta^\kappa$, we consider for all $\nu \in L^2_\rho(\boldsymbol{R}^n)$

$$(-1)^\kappa \varphi(x) \Delta^\kappa u(x) = \nu(x),\ x \in \boldsymbol{R}^n,$$

where

$$((-1)^\kappa \varphi \Delta^\kappa, w)_{L^2_\rho(\boldsymbol{R}^n)} = \int_{\boldsymbol{R}^n} \nabla^\kappa u \nabla^\kappa w dx,$$

without any boundary condition.

Remark 2.4 The operator $\Psi\Delta^\kappa$ is self-adjoint, symmetric, and strongly monotone in $L^2_\nu(\boldsymbol{R}^n)$.

The energy functional associated with the problem (1.1)-(1.2) is defined by

$$E(t) = \frac{1}{2}||u'||^2_{L^2_\rho(\boldsymbol{R}^n)} + \frac{1}{2}||\nabla^\kappa u||^2 - \frac{1}{p}||u||^p_{L^p_\rho(\boldsymbol{R}^n)}, \quad 0 \le t < t_{\max}, \tag{2.10}$$

and

$$E(0) = \frac{1}{2}||u_1||^2_{L^2_\rho(\boldsymbol{R}^n)} + \frac{1}{2}||\nabla^\kappa u_0||^2 - \frac{1}{p}||u_0||^p_{L^p_\rho(\boldsymbol{R}^n)}, \tag{2.11}$$

satisfy

$$E(t) + \int_0^t ||u'(s)||^q_{L^q_\rho(\boldsymbol{R}^n)} ds = E(0), \quad 0 \le t < t_{\max}. \tag{2.12}$$

We state the local existence result where its proof is similar to the techniques in [2, 15].

Theorem 2.5 If $u_0 \in D^{\kappa,2}(\boldsymbol{R}^n)$ and $u_1 \in L^2_\rho(\boldsymbol{R}^n)$ such that

$$(-1)^\kappa \varphi(x) \Delta^\kappa u_0 + \nu ||u_1||^{q-2}_{L^{q-2}_\rho(\boldsymbol{R}^n)} u_1 \in L^2(\boldsymbol{R}^n),$$

Suppose that $q > 2, p$ satisfies

$$\begin{cases} 2 \leq p < +\infty & if n = \kappa, 2\kappa \\ 2 \leq p \leq \dfrac{4\kappa - n}{n - 2\kappa} & if, n \geq 3\kappa. \end{cases} \tag{2.13}$$

then there exists $t_{\max} \leq +\infty$ such that (1.1)-(1.2) admit a unique generalized local solution u.

3. Finite Time Blow-up

In this section, we prove the blow-up results for the solution of problem (1.1)-(1.2) with the concavity method.

Theorem 3.1 Suppose that $q > 2$, p satisfies

$$2 \leq p \leq \frac{4\kappa - n}{n - 2\kappa} \; if, n \geq 3\kappa. \tag{3.1}$$

Then, for $E(0) = -r,\ r > 0$, the weak solution u cannot be extended to a maximal solution in $[0, t_{max})$ such that either $t_{max} = +\infty$, that is, the solution of the problem blows up for $0 < t_{\max} < s_0$, where s_0 is given in (3.15). i.e., the local solution satisfies

$$\lim_{t \to t_{\max}} \left(\| u' \|^2_{L^2_\rho(\mathbf{R}^n)} + \| \nabla^\kappa u \|^2 \right) = +\infty. \tag{3.2}$$

Proof 3.2 Let $t \in [0, t_{max}]$ and the constants $t_{max}, r, s > 0$ will be specified later. The concavity method is based on the construction and the properties of the functional

$$\alpha(t) = \int_{\boldsymbol{R}^n} \rho |u|^2 dx + \nu \, (\int_0^t ||u'||^{q-2}_{L^{q-2}_\rho(\boldsymbol{R}^n)} \int_{\boldsymbol{R}^n} \rho |u|^2 dxds + (t_{max} - t) \int_{\boldsymbol{R}^n} \rho |u_0|^2 dx) + r(t+s)^2 > 0. \tag{3.3}$$

Noting here that $\alpha(0) > 0$. Let

$$\beta(t) = (\int_{\boldsymbol{R}^n} \rho |u|^2 dx + \nu \int_0^t ||u'||^{q-2}_{L^{q-2}_\rho(\boldsymbol{R}^n)} \int_{\boldsymbol{R}^n} \rho |u|^2 dxds + r(t+s)^2)$$

$$\times (\int_{\boldsymbol{R}^n} \rho |u'|^2 dx + \nu \int_0^t \int_{\boldsymbol{R}^n} \rho |u'|^q dxds + r)$$

$$-(\int_{\boldsymbol{R}^n} \rho u u' dx + \nu \int_0^t ||u'||^{q-2}_{L^{q-2}_\rho(\boldsymbol{R}^n)} \int_{\boldsymbol{R}^n} \rho u u' dxds + r(t+s))^2. \tag{3.4}$$

Then

$$\alpha'(t) = 2 \int_{\boldsymbol{R}^n} \rho u u' dx + 2 ||u'||^{q-2}_{L^{q-2}_\rho(\boldsymbol{R}^n)} \nu (\int_0^t \int_{\boldsymbol{R}^n} \rho u u' dxds) + 2r(t+s), \tag{3.5}$$

and

$$\alpha''(t) = 2 \int_{\boldsymbol{R}^n} \rho u u'' dx + 2 \int_{\boldsymbol{R}^n} \rho u'^2 dx \; + 2 \, ||u'||^{q-2}_{L^{q-2}_\rho(\boldsymbol{R}^n)} \nu \left(\int_{\boldsymbol{R}^n} \rho u u' dx \right) + 2r \tag{3.6}$$

where we can choose s large enough to have $\alpha'(0) > 0$. Let u be the solution of (1.1)-(1.2). By multiplication of (1.1) by ρu and integrating over $\boldsymbol{R}^n$ to get

$$\int_{\boldsymbol{R}^n} \rho u u'' dx + \int_{\boldsymbol{R}^n} |\nabla^\kappa u|^2 \, dx + \nu \, ||u'||^{q-2}_{L^{q-2}_\rho(\boldsymbol{R}^n)} \cdot \frac{1}{2} \frac{d}{dt} \int_{\boldsymbol{R}^n} \rho |u|^2 \, dx = \int_{\boldsymbol{R}^n} \rho |u|^p \, dx \tag{3.7}$$

By (3.6), we have

$$\alpha''(t) = -2\int_{\boldsymbol{R}^n} |\nabla^\kappa u|^2\,dx + 2\int_{\boldsymbol{R}^n} \rho|u|^p\,dx + 2\int_{\boldsymbol{R}^n} \rho u'^2 dx + 2r \tag{3.8}$$

In addition, from (3.4) and (3.5), we have

$$\beta(t) = (\alpha(t) - \nu((t_{max} - t)\int_{\boldsymbol{R}^n} \rho|u_0|^2 dx))$$

$$\times (\int_{\boldsymbol{R}^n} \rho|u'|^2 dx + \nu\int_0^t \int_{\boldsymbol{R}^n} \rho|u'|^q dxds + r) - \frac{1}{4}\alpha'(t)^2,$$

or

$$\frac{1}{4}\alpha'(t)^2 = (\alpha(t) - \nu((t_{max} - t)\int_{\boldsymbol{R}^n} \rho|u_0|^2 dx))$$

$$\times (\int_{\boldsymbol{R}^n} \rho|u'|^2 dx + \nu\int_0^t \int_{\boldsymbol{R}^n} \rho|u'|^q dxds + r) - \beta(t).$$

Since

$$\frac{1}{4}\alpha'(t)^2 \le \alpha(t) \times (\int_{\boldsymbol{R}^n} \rho|u'|^2 dx + \nu\int_0^t \int_{\boldsymbol{R}^n} \rho|u'|^q dxds + r),$$

we have

$$\alpha(t)\alpha^{''\frac{p+2}{4}'2} \tag{3.9}$$

$$\alpha(t)\alpha''(t) - 2\left(\frac{p+2}{4}\right)\alpha'(t)^2 \ge \alpha(t)\left(\alpha''(t) - (p+2)\left(\int_{\boldsymbol{R}^n} \rho|u'|^2\,dx + \nu\int_0^t \int_{\boldsymbol{R}^n} \rho|u'|^q\,dx\,ds + r\right)\right)$$

On the other hand, by (2.12) and (3.6), we have

$$\begin{aligned}
&\alpha''(t) - (p+2)\left(\int_{\boldsymbol{R}^n} \rho|u'|^2\,dx + \nu\int_0^t \int_{\boldsymbol{R}^n} \rho|u'|^q\,dx\,ds + r\right) \\
&\ge -p(\int_{\boldsymbol{R}^n} \rho|u'|^2 dx + E(0) - E(t) - \frac{2}{p}\int_{\boldsymbol{R}^n} \rho|u|^p dx + r) - 2\int_{\boldsymbol{R}^n} |\nabla^\kappa u|^2 dx \\
&= -p(E(0) + r) + \frac{p-4}{2}\int_{\boldsymbol{R}^n} |\nabla^\kappa u|^2 dx.
\end{aligned} \tag{3.10}$$

Taking $r = -E(0) > 0$. It means with negative initial energy, (3.10) takes the form

$$\alpha''(t) - (p+2)\left(\int_{\boldsymbol{R}^n} \rho|u'|^2\,dx + \nu\int_0^t \int_{\boldsymbol{R}^n} \rho|u'|^q\,dx\,ds - E(0)\right) \ge \left(\frac{p-4}{2}\right)\int_{\boldsymbol{R}^n} |\nabla^\kappa u|^2 dx$$

Then (3.9) becomes

$$\alpha(t)\alpha''(t) - \left(\frac{p+2}{4}\right)\alpha'(t)^2 \ge \left(\frac{p+4}{2}\right)\alpha(t)\int_{\boldsymbol{R}^n} |\nabla^\kappa u|^2 dx$$

This ensures the concavity of the function α. In other words

$$\left(\alpha(t)^{\frac{2-p}{4}}\right)'' = \frac{p+2}{4}\,\alpha(t)^{\frac{-(p+6)}{4}}\left(\alpha(t)\alpha''(t) - \frac{p+2}{4}\,\alpha'(t)^2\right) \le 0 \qquad 3.11$$

We now choose t_{max} such that

$$t_{max} \geq \frac{4}{(p-2)} \frac{\alpha(0)}{\alpha'(0)}. \tag{3.12}$$

As the graph of any concave function lies below any tangent line of the function, we have

$$\alpha(t) \geq (\frac{4\alpha(0)^{\frac{p+2}{2}}}{4\alpha(0)-(p-2)\alpha'(0)t})^{\frac{4}{p-2}}, \tag{3.13}$$

so $\exists T \in (0, t_{max}]$ such that

$$\lim_{t \to T^-} (\int_{\mathbf{R}^n} \rho \,|\, u' \,|^2 \; dx + \nu \int_0^t \int_{\mathbf{R}^n} \rho \,|\, u' \,|^q \; dxds) = \infty.$$

At this stage, we found an upper bound for the finite time blow-up and then (3.2) is proved.

We search now the finite time t_{max}. By (3.12), for $t = 0$ in (3.3), (3.4), we have

$$T(s) = \frac{2(\int_{\boldsymbol{R}^n} \rho |u_0|^2 dx - E(0)s^2)}{(p-2)(\int_{\boldsymbol{R}^n} \rho u_0 u_1 dx + -E(0)s) - 2\nu \int_{\boldsymbol{R}^n} \rho |u_1|^q dx} \leq t_{max} \tag{3.14}$$

We choose the minimum value of $T(s)$. Since

$$T'(s) = \frac{2(p-2)E^2(0)s^2 + 4E(0)s[2\nu \int_{\boldsymbol{R}^n} \rho |u_1|^q dx - (p-2)\int_{\boldsymbol{R}^n} \rho u_0 u_1 dx] + 2(p-2)E(0)\int_{\boldsymbol{R}^n} \rho |u_0|^2 dx}{(p-2)(\int_{\boldsymbol{R}^n} \rho u_0 u_1 dx - E(0)s) - 2\nu \int_{\boldsymbol{R}^n} \rho |u_1|^q dx}.$$

Then, the minimum value of $T(s)$ in $(0, \infty)$ can be taken for

$$\begin{aligned} s_0 = \frac{-1}{E(0)(p-2)^2}(((2\nu \int_{\boldsymbol{R}^n} \rho |u_1|^q dx - (p-2)\int_{\boldsymbol{R}^n} \rho u_0 u_1 dx)^2 - (p-2)^2 E(0)\int_{\boldsymbol{R}^n} \rho |u_0|^2 dx)^{\frac{1}{2}} \\ +2\nu \int_{\boldsymbol{R}^n} \rho |u_1|^q dx - (p-2)\int_{\boldsymbol{R}^n} \rho u_0 u_1 dx) \end{aligned} \tag{3.15}$$

The proof of Theorem 3.1 is now completed.

Conflict of Interest

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