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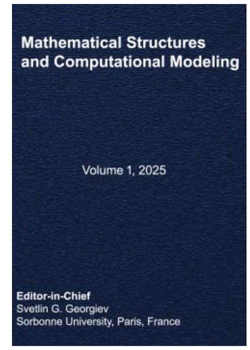

Existence of Solutions for the Fisher-Kolmogorov Equation

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ABSTRACT

In this paper we investigate the Cauchy problem for the Fisher-Kolmogorov equation for existence of global classical solutions. We give conditions under which the considered equation has at least one, at least two and at least three classical solutions. To prove our main results we propose a new approach based upon recent theoretical results.

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1. Introduction

In this paper, we investigate the Cauchy problem for the Fisher-Kolmogorov equation

$$\begin{aligned} u_t &= Du_{xx} + pu\left(1-\frac{u}{K}\right), \quad t>0, \quad x\in R, \\ u(0,x) &= u_0(x), \quad x\in R, \end{aligned} \tag{1}$$

where

D, K and p are constants, $K \neq 0$, $u_0 \in C^2(R)$ and $0 \leq u_0 \leq B$ on R for some nonnegative constant B.

The equation (1) is an one-dimensional reaction diffusion equation combining linear diffusion with a nonlinear logistic source term. The Fisher-Kolomogorov equation and its extensions have been used successfully in a wide range of applications including the study of spatial spreading of invasive species in ecology, in vitro cell biology experiments, in vivo malignant spreading, applications in combustion theory, in bush fire invasion.

In [6], the initial value problem (IVP) (1) is investigated in the case when u_0 is monotonic and continuous with $u_0(x)=1$ for $x<a$ and $u_0(x)=0$ for $x>b$, $-\infty<a<b<\infty$. In this paper the initial condition is arbitrary nonnegative bounded function. Thus, the results in this paper can be considered as complimentary results to the results in [6].

In this paper, under the conditions $(A1)$ we will investigate the equation (1) for existence of at least one solution, at least two nonnegative and at least three nonnegative solutions. For this aim, firstly it is given a new integral representation of the solutions of the considered problem and then they are constructed two operators so that any fixed point of their sum is a solution to the considered problem.

The paper is organized as follows. In the next section, we give some auxiliary results. In Section 3 we prove existence of at least one classical solution for the problem (1). In Section 4, we prove existence of at least two nonnegative classical solutions. In Section 5, we prove existence of at least two nonnegative classical solutions. In Section 6, we give an example to illustrate our main results.

2. Preliminary Results

Below, assume that X is a real Banach space. Now, we recall the definition for a completely continuous operator in a Banach space.

Definition 2.1 Let $K: M\subset X\to X$ be a map. We say that K is compact if $K(M)$ is contained in a compact subset of X. K is called a completely continuous map if it is continuous and it maps any bounded set into a relatively compact set.

The concept for k -set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 2.2. Let Ω_X be the class of all bounded sets of X. The Kuratowski measure of noncompactness $\alpha: \Omega_X \to [0,\infty)$ is defined by

$$\alpha(Y) = inf\left\{\delta>0: Y=\cup_{j=1}^m \quad Y_j \quad \text{and} \quad \mathrm{diam}(Y_j)\leq\delta, \quad j\in\{1,\dots,m\}\right\},$$

where $\mathrm{diam}(Y_j) = sup\{ Px-yP_X: x,y\in Y_j\}$ is the diameter of Y_j, $j\in\{1,\dots,m\}$.

For the main properties of measure of noncompactness we refer the reader to [2].

Definition 2.3. A mapping $K: X\to X$ is said to be k -set contraction if there exists a constant $k\geq 0$ such that

$$\alpha(K(Y)) \leq k\alpha(Y)$$

for any bounded set $Y\subset X$.

Obviously, if $K: X \to X$ is a completely continuous mapping, then K is 0-set contraction(see [4]).

Proposition 2.1. (Leray-Schauder nonlinear alternative [1]) Let $C \subset E$ be a convex, closed subset in a Banach space E, $0 \in U \subset C$ where U is an open set. Let $f: \overline{U} \to C$ be a continuous, compact map. Then

either f has a fixed point in $\overline{U}$,

or there exist $x \in \partial U$, and $\lambda \in (0,1)$ such that $x = \lambda f(x)$.

To prove our existence result we will use the following fixed point theorem. Its proof can be found in [5].

Theorem 2.1. Let E be a Banach space, Y a closed, convex subset of E, $0 \in Y$,

$$U = \{x \in Y:\ \|x\| < R\},$$

with $R > 0$. Consider two operators T and S, where

$$Tx = \varepsilon x,\ x \in \overline{U},$$

for $\varepsilon > 1$ and $S: \overline{U} \to E$ be such that

i. $I - S: \overline{U} \to Y$ continuous, compact and

ii. $\{x \in \overline{U}: x = \lambda(I - S)x, \quad \|x\| = R\} = \emptyset, \quad for\ any\ \ \lambda \in \left(0, \frac{1}{\varepsilon}\right).$

Then there exists $x^* \in \overline{U}$ such that

$$Tx^* + Sx^* = x^*.$$

Definition 2.4. Let X and Y be real Banach spaces. A map $K: X \to Y$ is called expansive if there exists a constant $h > 1$ for which one has the following inequality

$$\|Kx - Ky\|_Y \geq h\|x - y\|_X$$

for any $x, y \in X$.

Now, we will recall the definition for a cone in a Banach space.

Definition 2.5. A closed, convex set P in X is said to be cone if

1. $\alpha x \in P$ for any $\alpha \geq 0$ and for any $x \in P$,

2. $x, -x \in P$ implies $x = 0$.

Denote $P^* = P\backslash\{0\}$. The next result is a fixed point theorem which we will use to prove existence of at least two nonnegative global classical solutions of the IVP (1). For its proof, we refer the reader to [3], [8] and [7].

Theorem 2.2. Let P be a cone of a Banach space E; Ω a subset of P and $U_1, U_2\ and\ U_3$ three open bounded subsets of P such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T: \Omega \to P$ is an expansive mapping, $S: \overline{U}_3 \to E$ is a completely continuous map and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2\backslash\overline{U}_1) \cap \Omega \neq \emptyset, (U_3\backslash\overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in P^*$ such that the following conditions hold:

i. $Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,

ii. there exists $\varepsilon \geq 0$ such that $Sx \neq (I - T)(\lambda x)$, for all $\lambda \geq 1 + \varepsilon, x \in \partial U_2$ and $\lambda x \in \Omega$,

iii. $Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in P$ such that

$$x_1 \in \partial U_2 \cap \Omega \ and \ x_2 \in (\overline{U}_3 \backslash \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \backslash U_1) \cap \Omega \ and \ x_2 \in (\overline{U}_3 \backslash \overline{U}_2) \cap \Omega.$$

The following result will be used to prove the existence of three nonnegative solutions of our problem. For the proof, we use the same arguments used in [3] and [8].

Theorem 2.3. Let P be a cone of a Banach space E; Ω a subset of P and $U_1, U_2 \ and \ U_3$ three open bounded subsets of P such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T: \Omega \to E$ is an expansive mapping, $S: \overline{U}_3 \to E$ is a completely continuous one and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \backslash \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \backslash \overline{U}_2) \cap \Omega \neq \emptyset$, and there exist $w_0 \in P^*$ and $\varepsilon > 0$ small enough such that the following conditions hold:

i. $Sx \neq (I - T)(\lambda x)$, for all $\lambda \geq 1 + \varepsilon, x \in \partial U_1$ and $\lambda x \in \Omega$,

ii. $Sx \neq (I - T)(x - \lambda w_0)$, for all $\lambda \geq 0$ and $x \in \partial U_2 \cap (\Omega + \lambda w_0)$,

iii. $Sx \neq (I - T)(\lambda x)$, for all $\lambda \geq 1 + \varepsilon, x \in \partial U_3$ and $\lambda x \in \Omega$.

Then $T + S$ has at least three non trivial fixed points $x_1, x_2, x_3 \in P$ such that

$$x_1 \in \overline{U}_1 \cap \Omega \ and \ x_2 \in (U_2 \backslash \overline{U}_1) \cap \Omega \ and \ x_3 \in (\overline{U}_3 \backslash \overline{U}_2) \cap \Omega.$$

In $X = C^1([0, \infty), C^2(R))$ we introduce the norm

$$\|u\| = \{ \sup_{(t,x)\in[0,\infty)\times R} |u(t,x)|, \sup_{(t,x)\in[0,\infty)\times R} |u_t(t,x)|,$$

$$\sup_{(t,x)\in[0,\infty)\times R} |u_x(t,x)|, \sup_{(t,x)\in[0,\infty)\times R} |u_{xx}(t,x)|\},$$

provided it exists.

3. Existence of at Least One Solution

In this section, we will prove that the problem (1) has at least one solution.

For $u \in X$, define the operator

$$S_1(u)(t,x) = u(t,x) - u_0(x) - D \int_0^t u_{xx}(s,x) ds$$

$$-p \int_0^t u(s,x) \left(1 - \frac{u(s,x)}{K}\right) ds, \quad (t,x) \in [0,\infty) \times R.$$

Lemma 3.1. If $u \in X$ satisfies the equation

$$S_1(u)(t,x) = 0, \quad (t,x) \in [0,\infty) \times R, \tag{2}$$

then u is a solution to the problem (1).

Proof. By the equation (2), we get

$$\begin{aligned} 0 = \; & u(t,x) - u_0(x) - D \int_0^t u_{xx}(s,x) ds \\ & -p \int_0^t u(s,x) \left(1 - \frac{u(s,x)}{K}\right) ds, \quad (t,x) \in [0,\infty) \times R. \end{aligned} \tag{3}$$

We put $t = 0$ and we find

$$u(0,x) = u_0(x), \quad x \in R.$$

We differentiate the equation (3) with respect to t and we find

$$u_t(t,x) - Du_{xx}(t,x) - pu(t,x)\left(1 - \frac{u(t,x)}{K}\right) = 0, \quad (t,x) \in [0,\infty) \times R.$$

Thus, u is a solution to the problem (1). This completes the proof.

Let

$$B_1 = max\left\{2B, |D|B + |p|B\left(1 + \frac{B}{|K|}\right)\right\}.$$

Lemma 3.2. Suppose $(A1)$. If $u \in X$, $\|u\| \leq B$, then

$$|S_1(u)(t,x)| \leq B_1(1+t)(1+|x|), \quad (t,x) \in [0,\infty) \times R.$$

Proof. We have

$$\begin{aligned}
|S_1(u)(t,x)| &= |u(t,x) - u_0(x) - D\int_0^t \quad u_{xx}(s,x)ds \\
&\quad -p\int_0^t \quad u(s,x)\left(1 - \frac{u(s,x)}{K}\right)ds| \\
&\leq |u(t,x)| + |u_0(x)| + |D|\int_0^t \quad |u_{xx}(s,x)|ds \\
&\quad +|p|\int_0^t \quad |u(s,x)|\left(1 + \frac{|u(s,x)|}{|K|}\right)ds \\
&\leq 2B + |D|Bt + |p|B\left(1 + \frac{B}{|K|}\right)t \\
&\leq B_1(1+t) \\
&\leq B_1(1+t)(1+|x|), \quad (t,x) \in [0,\infty) \times R.
\end{aligned}$$

This completes the proof.

In addition, we suppose (A2) there exist a positive constant A and a function $g \in C([0,\infty) \times R)$, $g > 0$ on $(0,\infty) \times (R\backslash\{0\})$ with

$$g(0,x) = g(t,0) = 0, \quad (t,x) \in [0,\infty) \times R,$$

and

$$2(1+t)^2(1+|x|)(1+|x|+x^2)\left|\int_0^t \quad \int_0^x \quad g(s,y)dyds\right| \leq A, \quad (t,x) \in [0,\infty) \times R.$$

In the last section, we will give an example for a function g and a constant A that satisfy $(A2)$. For $u \in X$, define the operator

$$S_2(u)(t,x) = \int_0^t \quad \int_0^x \quad (t-s)(x-y)^2 g(s,y) S_1(u)(s,y)dyds, \quad (t,x) \in [0,\infty) \times R.$$

Lemma 3.3. Suppose $(A1)$ and $(A2)$. If $u \in X$ and $\|u\|P \leq B$, then

$$\|S_2 u\| \leq AB_1.$$

Proof. We have

$$\begin{aligned}
|S_2(u)(t,x)| &= |\int_0^t \quad \int_0^x \quad (t-s)(x-y)^2 g(s,y) S_1(u)(s,y)dyds| \\
&\leq |\int_0^t \quad \int_0^x \quad (t-s)(x-y)^2 g(s,y)|S_1(u)(s,y)|dyds| \\
&\leq B_1|\int_0^t \quad \int_0^x \quad (t-s)(x-y)^2(1+s)(1+|y|)g(s,y)dyds|
\end{aligned}$$

$$\leq B_1(1+t)(1+|x|)|\int_0^t \int_0^x (t-s)(|x|+|y|)^2 g(s,y)dyds|$$

$$\leq 2B_1 t(1+t)(1+|x|)|\int_0^t \int_0^x (|x|^2+|y|^2)g(s,y)dyds|$$

$$\leq 4B_1(1+t)^2(1+|x|)|x|^2|\int_0^t \int_0^x g(s,y)dyds|$$

$$\leq 4B_1(1+t)^2(1+|x|)(1+|x|+x^2)|\int_0^t \int_0^x g(s,y)dyds|$$

$$\leq AB_1, \quad (t,x)\in[0,\infty)\times R,$$

and

$$|\frac{\partial}{\partial x}S_2(u)(t,x)| = |2\int_0^t \int_0^x (t-s)(x-y)g(s,y)S_1(u)(s,y)dyds|$$

$$\leq 2|\int_0^t \int_0^x (t-s)(|x|+|y|)g(s,y)|S_1(u)(s,y)|dyds|$$

$$\leq 4B_1|x||\int_0^t \int_0^x (t-s)(1+s)(1+|y|)g(s,y)dyds|$$

$$\leq 4B_1(1+t)t|x|(1+|x|)|\int_0^t \int_0^x g(s,y)dyds|$$

$$\leq 4B_1(1+t)^2(1+|x|)(1+|x|+x^2)|\int_0^t \int_0^x g(s,y)dyds|$$

$$\leq AB_1, \quad (t,x)\in[0,\infty)\times R,$$

and

$$\left|\frac{\partial^2}{\partial x^2}S_2(u)(t,x)\right| = |2\int_0^t \int_0^x (t-s)g(s,y)S_1(u)(s,y)dyds|$$

$$\leq 2|\int_0^t \int_0^x (t-s)g(s,y)|S_1(u)(s,y)|dyds|$$

$$\leq 2B_1|\int_0^t \int_0^x (t-s)(1+s)(1+|y|)g(s,y)dyds|$$

$$\leq 2B_1(1+t)(1+|x|)|\int_0^t \int_0^x (t-s)g(s,y)dyds|$$

$$\leq 2B_1 t(1+t)(1+|x|)|\int_0^t \int_0^x g(s,y)dyds|$$

$$\leq 4B_1(1+t)^2(1+|x|)(1+|x|+x^2)|\int_0^t \int_0^x g(s,y)dyds|$$

$$\leq AB_1, \quad (t,x)\in[0,\infty)\times R,$$

and

$$\left|\frac{\partial}{\partial t}S_2(u)(t,x)\right| = |\int_0^t \int_0^x (x-y)^2 g(s,y)S_1(u)(s,y)dyds|$$

$$\leq |\int_0^t \int_0^x (x-y)^2 g(s,y)|S_1(u)(s,y)|dyds|$$

$$\leq B_1|\int_0^t \int_0^x (x-y)^2(1+s)(1+|y|)g(s,y)dyds|$$

$$\leq B_1(1+t)(1+|x|)|\int_0^t \int_0^x (|x|+|y|)^2 g(s,y)dyds|$$

$$\leq 2B_1(1+t)(1+|x|)|\int_0^t \int_0^x (|x|^2+|y|^2)g(s,y)dyds|$$

$$\leq 4B_1(1+t)^2(1+|x|)|x|^2|\int_0^t \int_0^x g(s,y)dyds|$$

$$\leq 4B_1(1+t)^2(1+|x|)(1+|x|+x^2)|\int_0^t \int_0^x g(s,y)dyds|$$

$$\leq AB_1, \quad (t,x) \in [0,\infty) \times R,$$

whereupon we get the desired result. This completes the proof.

Lemma 3.4. Suppose $(A1)$ and $(A2)$. If $u \in X$ satisfies the equation

$$S_2(u)(t,x) = C, \quad (t,x) \in [0,\infty) \times R, \tag{4}$$

for some constant C, then u is a solution to the problem (1).

Proof. We differentiate with respect to t and x the equation (4) and we find

$$g(t,x)S_1(u)(t,x) = 0, \quad (t,x) \in [0,\infty) \times R,$$

whereupon

$$S_1(u)(t,x) = 0, \quad (0,\infty) \times (R\backslash\{0\}).$$

Now, using that $S_1 u(\cdot,\cdot)$is a continuous function on $[0,\infty) \times R$, we find

$$\begin{aligned} 0 &= \lim_{t\to 0} S_1(u)(t,x) \\ &= S_1(u)(0,x) \\ &= \lim_{x\to 0} S_1(u)(t,x) \\ &= S_1(u)(t,0), \quad (t,x) \in [0,\infty) \times R. \end{aligned}$$

Hence, we conclude that u is a solution to the problem (1). This completes the proof.

Our main result in this section is as follows.

Theorem 3.1. Suppose $(A1)$ and $(A2)$. Then the equation (1) has at least one solution in X.

Proof. Let Y denote the set of all equi-continuous families in X with respect to the norm $P \cdot P$. Let also,

$$\tilde{Y} = \{u \in \tilde{\tilde{Y}}: u(t,x) \geq \tfrac{1}{2}\|u\|, \quad (t,x) \in [0,\infty) \times R\},$$

$Y = \overline{\tilde{Y}}$ be the closure of $\tilde{Y}$,

$$U = \{u \in Y: \|u\| < B\}.$$

For $u \in \overline{U}$ and $\varepsilon > 1$, define the operators

$$Tu(t,x) = \varepsilon u(t,x),$$

$$Su(t,x) = u(t,x) - \varepsilon u(t,x) - \varepsilon S_2(u)(t,x), \quad (t,x) \in [0,\infty) \times R.$$

For $u \in \overline{U}$, we have

$$\begin{aligned} \|(I-S)u\| &\leq \varepsilon\|u\| + \varepsilon\|S_2(u)\| \\ &\leq \varepsilon B + \varepsilon AB_1. \end{aligned}$$

Thus, $S: \overline{U} \to X$ is continuous and $(I-S)(\overline{U})$ resides in a compact subset of Y. Now, suppose that there is a $u \in \overline{U}$ so that $PuP = B$ and

$$u = \lambda(I-S)u$$

or

$$u = \lambda\varepsilon(u + S_2(u)), \tag{5}$$

for some $\lambda \in \left(0, \frac{1}{\varepsilon}\right)$. Then, using that $S_2(u)(0,x) = 0$, $x \in [0,\infty)$, and $\|u\| = B$, we get $u(0,x) \geq \frac{B}{2}$, $x \in [0,\infty)$, and

$$u(0,x) = \lambda\varepsilon(u(0,x) - S_2(u)(0,x)) = \lambda\varepsilon u(0,x), \quad x \in [0,\infty),$$

whereupon $\lambda\varepsilon = 1$, which is a contradiction. Consequently

$$\{u \in \overline{U}: u = \lambda_1(I - S)u,\ PuP = B\} = \emptyset$$

for any $\lambda_1 \in \left(0, \frac{1}{\varepsilon}\right)$. Then, from Theorem 2.1, it follows that the operator $T + S$ has a fixed point $u^* \in Y$. Therefore

$$u^*(t,x) = Tu^*(t,x) + Su^*(t,x)$$
$$= \varepsilon u^*(t,x) + u^*(t,x) - \varepsilon u^*(t,x) - \varepsilon S_2(u^*)(t,x), \quad (t,x) \in [0,\infty) \times R,$$

whereupon

$$S_2(u^*)(t,x) = 0, \quad (t,x) \in [0,\infty) \times R.$$

From here, u^* is a solution to the problem (1). From here and from Lemma 3.4, it follows that u is a solution to the equation (1). This completes the proof.

4. Existence of at Least Two Solutions

Let X be the space used in the previous section and r, L and R_1 be positive constants such that

(A3) $r < L < R_1$.

Our main result in this section is as follows.

Theorem 4.1 Suppose that $(A1)$, $(A2)$ and $(A3)$ hold. Then the equation (1) has at least two nonnegative solutions in X.

Proof. Let

$$\tilde{P} = \{u \in X: u \geq 0 \quad \text{on} \quad [0,\infty) \times \mathrm{R}\}.$$

With P we will denote the set of all equi-continuous families in $\tilde{P}$. For $v \in X$, define the operators

$$T_1 v(t,x) = (1 + m\varepsilon)v(t,x),$$
$$S_3 v(t,x) = -\varepsilon S_2(v)(t,x) - m\varepsilon v(t,x) - \varepsilon A B_1,$$

$(t,x) \in [0,\infty) \times R$. Note that any fixed point $v \in X$ of the operator $T_1 + S_3$ is a solution to the equation (1). Define

$$\Omega = P,$$
$$U_1 = P_r = \{v \in P: \|v\| < r\},$$
$$U_2 = P_L = \{v \in P: \|v\| < L\},$$
$$U_3 = P_{R_1} = \{v \in P: \|v\| < R_1\}.$$

1. For $v_1, v_2 \in \Omega$, we have

$$\|T_1 v_1 - T_1 v_2\| = (1 + m\varepsilon)\|v_1 - v_2\|,$$

whereupon $T_1: \Omega \to X$ is an expansive operator with a constant $h = 1 + m\varepsilon > 1$.

2. For $v \in \overline{P}_{R_1}$, we get

$$\begin{aligned} \|S_3 v\| &\leq \varepsilon \|S_2(v)\| + m\varepsilon \|v\| + \varepsilon AB_1 \\ &\leq \varepsilon(2AB_1 + mR_1). \end{aligned}$$

Therefore $S_3(\overline{P}_{R_1})$ is uniformly bounded. Since $S_3: \overline{P}_{R_1} \to X$ is continuous, we have that $S_3(\overline{P}_{R_1})$ is equi-continuous. Consequently $S_3: \overline{P}_{R_1} \to X$ is a 0-set contraction.

3. Let $v_1 \in \overline{P}_{R_1}$. Set

$$v_2 = v_1 + \frac{1}{m} S_2(v_1) + \frac{1}{m} AB_1.$$

We have $v_2 \geq 0$ on $[0, \infty) \times R$. Therefore $v_2 \in \Omega$ and

$$-\varepsilon m v_2 = -\varepsilon m v_1 - \varepsilon S_2(v_1) - \varepsilon AB_1$$

or

$$\begin{aligned} (I - T_1) v_2 &= -\varepsilon m v_2 \\ &= S_3 v_1. \end{aligned}$$

Consequently $S_3(\overline{P}_{R_1}) \subset (I - T_1)(\Omega)$.

4. Assume that for any $v_0 \in P^*$ there exist $\lambda \geq 0$ and $v \in \partial P_r \cap (\Omega + \lambda v_0)$ or $v \in \partial P_{R_1} \cap (\Omega + \lambda v_0)$ such that

$$S_3 v = (I - T_1)(v - \lambda v_0).$$

Then

$$-\varepsilon |S_2(v)| - \varepsilon AB_1 - m\varepsilon v = -m\varepsilon(v - \lambda v_0)$$

or

$$-|S_2(v)| - AB_1 = \lambda m v_0.$$

This is a contradiction.

5. Let $\varepsilon_1 = \frac{3AB_1}{mL}$. Suppose that there exist a $v_1 \in \partial P_L$ and $\lambda_1 \geq 1 + \varepsilon_1$ such that

$$S_3 v_1 = (I - T_1)(\lambda_1 v_1). \tag{6}$$

Moreover,

$$-\varepsilon |S_2(v_1)| - \varepsilon AB_1 - m\varepsilon v_1 = -\lambda_1 m \varepsilon v_1,$$

or

$$|S_2(v_1)| + AB_1 + m v_1 = \lambda_1 m v_1.$$

From here,

$$\begin{aligned} \lambda_1 mL &= \lambda_1 m \|v_1\| \\ &\leq \|S_2(v_1)\| + m\|v_1\| + AB_1 \\ &\leq 2AB_1 + mL \end{aligned}$$

and

$$\frac{2AB_1}{mL} + 1 \geq \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 2.2 hold. Hence, the problem (1) has at least two solutions u_1 and u_2 so that

$$\|u_1\| = L < \|u_2\| < R_1$$

or

$$r < \|u_1\| < L < \|u_2\| < R_1.$$

5. Existence of at Least Three Solutions

Our main result for existence of at least three solutions of the problem (1) is as follows.

Theorem 5.1 Under the hypotheses $(A1)$, $(A2)$ and $(A3)$, the problem (1) has at least three nonnegative solutions $u_1, u_2, u_3 \in X$.

Proof.

1. Let $\eta = \frac{3AB_1}{mL}$. Assume that there are $\lambda_1 \geq 1 + \eta$, $u \in \partial U_1$ and $\lambda_1 u \in \Omega$ so that

$$S_3(u) = (I - T_1)(\lambda_1 u).$$

Then

$$-\varepsilon|S_2(u)| - \varepsilon AB_1 - m\varepsilon u = -m\varepsilon\lambda_1 u$$

or

$$|S_2(u)| + AB_1 + mu = \lambda_1 mu.$$

Hence,

$$\begin{aligned}\lambda_1 mL &= \lambda_1 m\|u\| \\ &\leq \|S_2(u)\| + m\|u\| + AB_1 \\ &\leq 2AB_1 + mL,\end{aligned}$$

whereupon

$$\lambda_1 \leq 1 + \frac{2AB_1}{mL},$$

which is a contradiction. Thus, the condition (i) of Theorem 2.3 holds.

2. Now, assume that there are $\lambda_1 \geq 1 + \eta$, $u \in \partial U_3$ and $\lambda_1 u \in \Omega$ so that

$$S_3(u) = (I - T_1)(\lambda_1 u).$$

As above,

$$\begin{aligned}\lambda_1 mR_1 &= \lambda_1 m\|u\| \\ &\leq \|S_2(u)\| + m\|u\| + AB_1 \\ &\leq 2AB_1 + mR_1,\end{aligned}$$

whereupon

$$\lambda_1 \le 1 + \frac{2AB_1}{mR_1} \le 1 + \frac{2AB_1}{mL},$$

which is a contradiction. Hence, the condition (iii) of Theorem 2.3 holds.

3. Assume that for any $u_0 \in P^*$ there exist $\lambda_1 \ge 0$ and $u \in \partial P_L \cap (\Omega + \lambda_1 u_0)$ such that

$$S_3(u) = (I - T_1)(u - \lambda_1 u_0).$$

Then

$$-\varepsilon|S_2(u)| - \varepsilon AB_1 - m\varepsilon u = -m\varepsilon(u - \lambda_1 u_0)$$

or

$$-|S_2(u)| - AB_1 = \lambda_1 m u_0.$$

This is a contradiction. Form here, the condition (ii) of Theorem 2.3 holds.

Now, by Theorem 2.3, it follows that the problem (1) has at least three classical solutions u_1, u_2 and u_3 such that

$$u_1 \in \partial U_1 \cap \Omega \ and \ u_2 \in (U_2 \backslash \overline{U}_1) \cap \Omega \ and \ u_3 \in (\overline{U}_3) \backslash \overline{U}_2) \cap \Omega,$$

or

$$u_1 \in U_1 \cap \Omega \ and \ u_2 \in (U_2 \backslash \overline{U}_1) \cap \Omega \ and \ u_3 \in (\overline{U}_3) \backslash \overline{U}_2) \cap \Omega.$$

6. An Example

Below, we will illustrate our main results. Take

$$h(s) = log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}, \quad l(s) = arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}, \quad s \in \ R, \quad s \ne \pm 1.$$

Then

$$h'(s) = \frac{22\sqrt{2}s^{10}(1-s^{22})}{(1-s^{11}\sqrt{2}+s^{22})(1+s^{11}\sqrt{2}+s^{22})},$$

$$l'(s) = \frac{11\sqrt{2}s^{10}(1+s^{22})}{1+s^{44}}, \quad s \in \ R, \quad s \ne \pm 1.$$

Therefore

$$-\infty < \lim_{s\to\pm\infty}(1+s+s^2)h(s) < \infty,$$

$$-\infty < \lim_{s\to\pm\infty}(1+s+s^2)l(s) < \infty.$$

Hence, there exists a positive constant D_1 so that

$$(1+|s|)^3(1+|s|+s^2)\left(\frac{1}{44\sqrt{2}}log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}} + \frac{1}{22\sqrt{2}}arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right) \quad \le \quad D_1,$$

$s \in R$. Note that $\lim_{s\to\pm 1} l(s) = \frac{\pi}{2}$ and by [9] (pp. 707, Integral 79), we have

$$\int\frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}}\log\frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2} + \frac{1}{2\sqrt{2}}\arctan\frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1+s^{44})(1+s+s^2)^2}, \quad s \in \mathrm{R},$$

and

$$g_1(t,x) = Q(t)Q(x), \quad (t,x) \in [0,\infty) \times \mathrm{R}.$$

Then there exists a constant $D_1 > 0$ such that

$$(1+t)^2(1+|x|)(1+|x|+x^2)\left|\int_0^t \int_0^x g_1(\tau,y)d\tau dy\right| \le D_1,$$

$(t,x) \in [0,\infty) \times \mathrm{R}$. Let

$$g(t,x) = \frac{A}{D_1} g_1(t,x), \quad (t,x) \in [0,\infty) \times \mathrm{R},$$

Then

$$(1+t)^2(1+|x|)(1+|x|+x^2)\left|\int_0^t \int_0^x g(\tau,y)d\tau dy\right| \le A,$$

$(t,x) \in [0,\infty) \times R$, i.e., $(A2)$ holds. Let

$$R_1 = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad B = p = K = 1, \quad D = 2$$

and

$$A = \frac{1}{10B_1}, \quad \varepsilon = \frac{1}{8}.$$

Then $B_1 = 4$, $A = \frac{1}{40}$ and

$$r < L < R_1,$$

i.e., $(A3)$ holds.

Let

$$u_0(x) = \frac{1}{1+x^2}, \quad x \in \mathrm{R}.$$

Therefore for the problem

$$u_t = 2u_{xx} + u(1-u), \quad t > 0, \quad x \in R,$$
$$u(0,x) = \frac{1}{1+x^2}, \quad x \in R,$$

are fulfilled all conditions of Theorem 3.1, Theorem 4.1 and Theorem 5.1.

Conflict of Interest

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