

Mathematical Structures and Computational Modeling

ISSN (online): xxxx-xxxx

Mathematical Structures
and Computational Modeling

Volume 1, 2025

Editor-in-Chief
Svetlin G. Georgiev
Sorbonne University, Paris, France

A Coupled Majda Biello Type System with Damping

Aissa Boukarou^{id 1,*} and Mohamad Biomy²

¹Aissa Boukarou: University of Science and Technology Houari Boumediene, Algiers, Algeria

²Department of Management Information Systems, College of Business Administration, Qassim University, Buraydah 52571, Saudi Arabia

ARTICLE INFO

Article Type: Research Article

Keywords:

Damping

Well-posedness

Conservation law

Majda Biello system

Korteweg-de Vries equation

Timeline:

Received: June 05, 2025

Accepted: July 08, 2025

Published: July 12, 2025

Citation: Boukarou A, Biomy M. A coupled Majda Biello type system with damping. Math Struct Comput Model. 2025; 1: 60-72.

DOI: <https://doi.org/xx.xxxxx/xxxx-xxxx.2025.1.1>

ABSTRACT

In this paper, we study a Majda Biello Type system with weak damping terms and $(u_0, v_0) \in L^\sigma(\mathbb{R}) \times L^\sigma(\mathbb{R})$.

$$\begin{cases} u_t + u_{xxx} + \beta_1 uu_x + \beta_2 uv_x + \beta_3 u_x v + \beta_4 vv_x = 0, \\ v_t + \alpha v_{xxx} + \beta_5 uu_x + \beta_6 uv_x + \beta_7 u_x v + \beta_8 vv_x = 0. \end{cases}$$

The system involves two functions, $d_1(x)$ and $d_2(x)$, which serve as damping coefficients. We first establish local well-posedness by applying the fixed point theorem in Fourier restriction spaces $Y_{\sigma,k}(\mathbb{R}^2) \times Y_{\sigma,k}^\alpha(\mathbb{R}^2)$. Subsequently, using an approximate conservation law, we extend the local well-posedness result to a global one, demonstrating that the radius of analyticity of the solutions remains uniformly bounded below by a fixed positive constant for all time.

[2010] Mathematics Subject Classification: 35L70, 35L05, 35B35.

*Corresponding Author

Email: boukarouaissa@gmail.com

Tel: +(213) 669968854

1. Introduction

In recent years, periodic KdV systems have been actively studied both quantitatively and qualitatively. The significance of the present research lies primarily in KdV systems of the Majda Biello type. A comprehensive study of solutions in spaces of analytic functions facilitates applications in various scientific fields.

We consider the Majda Biello system, given by

$$\begin{cases} u_t + u_{xxx} + vv_x = 0 \\ v_t + \alpha v_{xxx} + (uv)_x = 0 \\ (u, v)|_{t=0} = (u_0, v_0), \end{cases} \quad (x, t) \in T_\lambda \times \mathbb{R} \text{ or } \mathbb{R} \times \mathbb{R}, \quad (1.1)$$

where $T_\lambda = [0, 2\pi\lambda]$ with $\lambda \geq 1, 0 < \alpha \leq 1$, and u and v are real-valued functions.

In [9], Oh studied the local well-posedness of the Majda Biello system in Sobolev spaces $H^s(\mathbb{R})$, in both periodic and non-periodic settings. Elmansouri et al. [10] improved this result in the periodic case by establishing local well-posedness in analytic Gevrey spaces $G_{\sigma, \delta, s}(T_\gamma)$ and analyzing the regularity of the solutions.

$$G^{\sigma, \delta, s}(T_\gamma) = \left\{ g \in L^2(T_\gamma); \|g\|_{G^{\sigma, \delta, s}(T_\gamma)}^2 = \sum_{k \in \mathbb{Z}} e^{2\delta|k|^{1/\sigma}} \langle k \rangle^{2s} |F_x g(k)|^2 d\xi < \infty \right\}$$

This work focuses on the Cauchy problem for a Majda—Biello-type system with a damping term $d_i(x)$, $i = 1, 2$

$$\begin{cases} u_t + u_{xxx} + \beta_1 uu_x + \beta_2 uv_x + \beta_3 u_x v + \beta_4 vv_x + d_1(x)u = 0, \\ v_t + \alpha v_{xxx} + \beta_5 uu_x + \beta_6 uv_x + \beta_7 u_x v + \beta_8 vv_x + d_2(x)v = 0, \\ (u, v)|_{t=0} = (u_0, v_0). \end{cases} \quad 0 < \alpha < 1, \quad (1.2)$$

The author in [6] proved that the initial value problem (1.2) is locally and globally well-posed in analytic Gevrey spaces $G^\sigma(\mathbb{R}) \times G^\sigma(\mathbb{R})$.

$$G^\sigma(\mathbb{R}) = \left\{ g \in L^2(\mathbb{R}); \|g\|_{G^\sigma(\mathbb{R})}^2 = \int_{\mathbb{R}} \exp(2\sigma|\rho|) |F_x g(\rho)|^2 d\rho < \infty, \sigma \geq 0 \right\}. \quad (1.3)$$

In this paper, we follow the approach of the authors in [2, 4, 8, 11] and choose the most suitable analytic function space for our system, defined by

$$L^\sigma(\mathbb{R}) = \left\{ g \in L^2(\mathbb{R}); \|g\|_{L^\sigma(\mathbb{R})}^2 = \int_{\mathbb{R}} (\cosh(\sigma|\rho|))^2 |F_x(g(\rho))|^2 d\rho < \infty, \sigma \geq 0 \right\}. \quad (1.4)$$

Since

$$\exp(\sigma|\rho|) \sim \cosh(\sigma|\rho|),$$

we have

$$\|\cdot\|_{L^\sigma(\mathbb{R})}^2 \sim \|\cdot\|_{G^\sigma(\mathbb{R})}^2.$$

For $\sigma = 0$, we have $L^0(\mathbb{R})$ reduced $L^2(\mathbb{R})$. We state the embedding properties of analytic spaces with $0 < \sigma' < \sigma$,

$$L^\sigma(\mathbb{R}) \subset L^{\sigma'}(\mathbb{R}), \quad (1.5)$$

i.e.

$$\|f\|_{L^{\sigma'}} \lesssim_{\sigma, \sigma'} \|f\|_{L^\sigma}.$$

We need to impose some assumptions on the damping coefficient $d_i(x)$, $i = 1, 2$.

- Damping effect i.e. there exists $T > 0$ so that

$$d_i(x) \geq T > 0, \quad \text{for all } x \in \mathbb{R}.$$

- Analyticity i.e. there exist constants $M_1, M_2 > 0$ so that

$$\|\partial_x^j d_i(x)\|_{L^\infty(\mathbb{R})} \leq M_1 M_2^j (j!), \quad \text{for all } j \in \mathbb{N}.$$

Notation: First, we introduce some notations and function spaces used in this paper. We use $A \lesssim_{c_1, c_2, \dots} B$ to denote $A \leq CB$ with some constant C depending on $c_1, c_2, \dots$. If C is an absolute constant, we shall write $A \lesssim B$.

If both $A \lesssim_{c_1, c_2, \dots} B$ and $B \lesssim_{c_1, c_2, \dots} A$ hold, then we write $A \sim_{c_1, c_2, \dots} B$.

We introduce the following operators:

$$\begin{aligned}\Delta_1 &= \frac{1}{2} \partial_x [(cosh(\sigma|\Lambda|)u)^2 - cosh(\sigma|\Lambda|)(u^2)] \\ \Delta_2 &= \frac{1}{2} \partial_x [(cosh(\sigma|\Lambda|)v)^2 - cosh(\sigma|\Lambda|)(v^2)] \\ \Theta_1 &= [(cosh(\sigma|\Lambda|)u) \partial_x (cosh(\sigma|\Lambda|)v) - cosh(\sigma|\Lambda|)(u \partial_x v)] \\ \Theta_2 &= [(cosh(\sigma|\Lambda|)v) \partial_x (cosh(\sigma|\Lambda|)u) - cosh(\sigma|\Lambda|)(v \partial_x u)] \\ \Gamma_1 &= [d_1 cosh(\sigma|\Lambda|)u - cosh(\sigma|\Lambda|)(d_1 u)] \\ \Gamma_2 &= [d_2 cosh(\sigma|\Lambda|)v - cosh(\sigma|\Lambda|)(d_2 v)],\end{aligned}$$

where $\Lambda = -i\partial_x$ is a Fourier multiplier with symbol $|\rho|$.

The spatial Fourier transform is denoted by

$$F_x(g) = \hat{g}(\rho) = \int_{\mathbb{R}} \exp(-ix\rho)g(x) dx,$$

and

$$F_{x,t}(u) = \tilde{u}(\rho, \tau) = \int_{\mathbb{R}^2} \exp(-i(x\rho + t\tau))u(x, t) dxdt,$$

is the space-time Fourier transform.

Now, we shall give an elementary space which will be used in the present paper.

Fourier restriction spaces: The hybrid space between the analytic space and a space of the Fourier restriction is considered for $\kappa \in \mathbb{R}$ and $\sigma > 0$ where $Y_{\sigma, \kappa}(\mathbb{R}^2)$ is a Banach space such that

$$\begin{aligned}\|u\|_{Y_{\sigma, \kappa}(\mathbb{R}^2)}^2 &= \int_{\mathbb{R}^2} (cosh(\sigma|\rho|))^2 (1 + |\tau - \rho^3|)^{2\kappa} |F_{x,t}(u(\rho, \tau))|^2 d\rho d\tau, \\ \|v\|_{Y_{\sigma, \kappa}^{\alpha}(\mathbb{R}^2)}^2 &= \int_{\mathbb{R}^2} (cosh(\sigma|\rho|))^2 (1 + |\tau - \alpha\rho^3|)^{2\kappa} |F_{x,t}(v(\rho, \tau))|^2 d\rho d\tau.\end{aligned}$$

Also, for $\delta > 0$, $Y_{\sigma, \kappa}^T(\mathbb{R}^2)$ represents the restricted analytic space given by

$$\|u\|_{Y_{\sigma, \kappa}^{\delta}(\mathbb{R}^2)} = \inf \left\{ \|U\|_{Y_{\sigma, \kappa}(\mathbb{R}^2)} : U = u \text{ on } (0, \delta) \times \mathbb{R} \right\}, \quad (1.6)$$

and $Y_{\sigma, \kappa}^{\alpha, \delta}(\mathbb{R}^2)$

$$\|v\|_{Y_{\sigma, \kappa}^{\alpha, \delta}(\mathbb{R}^2)} = \inf \left\{ \|V\|_{Y_{\sigma, \kappa}^{\alpha}(\mathbb{R}^2)} : V = v \text{ on } (0, \delta) \times \mathbb{R} \right\}. \quad (1.7)$$

We are now ready to derive certain estimates in the Banach spaces $A^{\sigma}(\mathbb{R})$, $\sigma > 0$, where

$$\|g\|_{A^{\sigma}(\mathbb{R})} = \sum_{i \geq 0} (i+1)^{\frac{1}{4}} \frac{\sigma^i}{i!} \|\partial_x^i g\|_{L^{\infty}(\mathbb{R})}.$$

Finally, we define the spaces

$$\begin{aligned}\mathbb{Y}_{\alpha}^{\sigma, \kappa}(\mathbb{R}^2) &= Y_{\alpha}^{\sigma, \kappa}(\mathbb{R}^2) \times Y^{\sigma, \kappa}(\mathbb{R}^2), \\ \mathbb{L}^{\sigma}(\mathbb{R}) &= L^{\sigma}(\mathbb{R}) \times L^{\sigma}(\mathbb{R}), \\ \mathbb{L}^2(\mathbb{R}) &= L^2(\mathbb{R}) \times L^2(\mathbb{R})\end{aligned}$$

$$\mathbb{L}^\infty(\mathbb{R}) = L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R}) \quad \text{and} \quad \mathbb{A}^\sigma(\mathbb{R}) = A^\sigma(\mathbb{R}) \times A^\sigma(\mathbb{R}),$$

with norms

$$\begin{aligned} \|(u, v)\|_{\mathbb{Y}_\alpha^{\sigma, \kappa}} &= \max\{\|u\|_{Y^{\sigma, \kappa}}, \|v\|_{Y_\alpha^{\sigma, \kappa}}\}, \\ \|(u_0, v_0)\|_{\mathbb{L}^\sigma} &= \max\{\|u_0\|_{L^\sigma}, \|v_0\|_{L^\sigma}\}, \\ \|(\Gamma_1, \Gamma_2)\|_{\mathbb{L}^2} &= \max\{\|\Gamma_1\|_{L^2}, \|\Gamma_2\|_{L^2}\}, \\ \|(\mathbf{d}_1, \mathbf{d}_2)\|_{\mathbb{L}^\infty} &= \max\{\|\mathbf{d}_1\|_{L^\infty}, \|\mathbf{d}_2\|_{L^\infty}\}. \end{aligned}$$

and

$$\|(\mathbf{d}_1, \mathbf{d}_2)\|_{\mathbb{A}^\sigma} = \max\{\|\mathbf{d}_1\|_{A^\sigma}, \|\mathbf{d}_2\|_{A^\sigma}\}.$$

We will show that $Y_\alpha^{\sigma, \kappa}(\mathbb{R}^2)$ is continuously embedded in

$$C([0, \delta], L^\sigma(\mathbb{R})) \times C([0, \delta], L^\sigma(\mathbb{R})), \quad \kappa > \frac{1}{2}.$$

Lemma 1.1 Let $\kappa > \frac{1}{2}$ and $\sigma > 0$, Then,

$$\forall (u, v) \in \mathbb{Y}_\alpha^{\sigma, \kappa}(\mathbb{R}^2),$$

we have

$$|(u, v)|_{\delta, \alpha} = \sup_{t \in [0, \delta]} \|(u(\cdot, t), u(\cdot, t))\|_{\mathbb{L}^\sigma(\mathbb{R})} \lesssim_\kappa \|(u, v)\|_{\mathbb{Y}_\alpha^{\sigma, \kappa}(\mathbb{R}^2)},$$

2. Linear Estimates

Owing to Duhamel's formula, the solution of (1.2) can be written as

$$\begin{cases} u(x, t) = W(t)u_0(x) + \int_0^t W(t-r)(\mathbf{d}_1 u + N_1)(x, r)dr \\ v(x, t) = W_\alpha(t)v_0(x) + \int_0^t W_\alpha(t-r)(\mathbf{d}_2 v + N_2)(x, r)dr. \end{cases}$$

where

$$N_1 = \beta_1 uu_x + \beta_2 uv_x + \beta_3 u_x v + \beta_4 vv_x, \quad N_2 = \beta_5 uu_x + \beta_6 uv_x + \beta_7 u_x v + \beta_8 vv_x.$$

Here $W(t)$ and $W_\alpha(t)$ denote respectively the solution groups associated with the homogeneous linear problems:

$$\begin{cases} \partial_t u + \partial_x^3 u = 0 \\ u(x, 0) = u_0(x), \end{cases}$$

and

$$\begin{cases} \partial_t v + \alpha \partial_x^3 v = 0 \\ v(x, 0) = v_0(x), \end{cases}$$

where

$$W(t) = F_x^{-1} e^{-it\rho^3} F_x, \quad \text{and} \quad W_\alpha(t) = F_x^{-1} e^{-it\alpha\rho^3} F_x.$$

So we define two operators (Φ, Ψ) , such that:

$$\begin{aligned} (\Phi(u), \Psi(v)) &= (W(t)u_0(x) + \int_0^t W(t-r)(\mathbf{d}_1 u + N_1)(x, r)dr, \\ &\quad W_\alpha(t)v_0(x) + \int_0^t W_\alpha(t-r)(\mathbf{d}_2 v + N_2)(x, r)dr). \end{aligned}$$

By using a cut-off function $\psi \in C_0^\infty(\mathbb{R})$, with $\psi = 1$ in $[-\frac{1}{2}, \frac{1}{2}]$, $\text{supp } \psi \subset [-1, 1]$ and $\psi_\delta(t) = \psi(\frac{t}{\delta})$, we may localize it in time. Define the operator $(\Phi_1(u), \Psi_1(v))$ given by

$$(\Phi(u), \Psi(v)) = (\psi_1 W(t)u_0(x) + \int_0^t W(t-r)(d_1 u + \psi_1 N_1)(x, r)dr, \psi_1 W_\alpha(t)v_0(x) + \psi_1 \int_0^t W_\alpha(t-r)(d_2 v + \psi_1 N_2)(x, r)dr). \quad (2.1)$$

defined for system (1.2). We are going to solve $(\Phi(u), \Psi(v)) = (u, v)$.

Lemma 2.1. ([7]) Let $\sigma > 0$ and $\frac{1}{2} < \kappa < 1$, we have

$$\begin{aligned} \|\psi_1(t)W(t)u_0(x)\|_{Y^{\sigma, \kappa}(\mathbb{R}^2)} &\lesssim_{\psi_1} \|u_0(x)\|_{L^\sigma(\mathbb{R})}, \\ \|\psi_1(t)W(t)v_0(x)\|_{Y_\alpha^{\sigma, \kappa}(\mathbb{R}^2)} &\lesssim_{\psi_1} \|v_0(x)\|_{L^\sigma(\mathbb{R})}. \\ \left\| \psi_1(t) \int_0^t W(t-r)N_1(x, r)dr \right\|_{Y^{\sigma, \kappa}(\mathbb{R}^2)} &\lesssim_{\psi_1} \|N_1(x, t)\|_{Y^{\sigma, \kappa-1}(\mathbb{R}^2)}, \\ \left\| \psi_1(t) \int_0^t W_\alpha(t-r)N_2(x, r)dr \right\|_{Y_\alpha^{\sigma, \kappa}(\mathbb{R}^2)} &\lesssim_{\psi_1} \|N_2(x, t)\|_{Y_\alpha^{\sigma, \kappa-1}(\mathbb{R}^2)}. \end{aligned}$$

If $\frac{1}{2} < \kappa \leq \kappa' < 1$, then $\forall \delta > 0$, we have

$$\|\psi_\delta(t)N_1(x, t)\|_{Y^{\sigma, \kappa-1}(\mathbb{R}^2)} \lesssim_{\kappa, \kappa'} \delta^{\kappa'-\kappa} \|N_1(x, t)\|_{Y^{\sigma, \kappa'-1}(\mathbb{R}^2)}, \quad (2.2)$$

$$\|\psi_\delta(t)N_2(x, t)\|_{Y_\alpha^{\sigma, \kappa-1}(\mathbb{R}^2)} \lesssim_{\kappa, \kappa'} \delta^{\kappa'-\kappa} \|N_2(x, t)\|_{Y_\alpha^{\sigma, \kappa'-1}(\mathbb{R}^2)}. \quad (2.3)$$

Lemma 2.2. ([8]) If $-\frac{1}{2} < \kappa < \frac{1}{2}$, then for $I \subset [0, \delta]$

$$\begin{aligned} \|\chi_I u\|_{Y^{\sigma, \kappa}} &\lesssim_\kappa \|u\|_{Y_\delta^{\sigma, \kappa}}, \\ \|\chi_I v\|_{Y_\alpha^{\sigma, \kappa}} &\lesssim_\kappa \|v\|_{Y_{\alpha, \delta}^{\sigma, \kappa}}, \end{aligned}$$

where χ_I is the characteristic function in I .

Lemma 2.3. ([11]) For any $\sigma > 0$ we have

$$\begin{aligned} \|\chi_{[0, \delta]}(t)e^{-2T(\delta-t)}u\|_{Y_\delta^{\sigma, 1-\kappa}} &\lesssim \|u\|_{Y_\delta^{\sigma, \kappa}}, \\ \|\chi_{[0, \delta]}(t)e^{-2T(\delta-t)}v\|_{Y_{\alpha, \delta}^{\sigma, 1-\kappa}} &\lesssim \|v\|_{Y_{\alpha, \delta}^{\sigma, \kappa}}. \end{aligned} \quad (2.4)$$

Lemma 2.4. ([11]) For any $\sigma > 0, d_1, d_2 \in A^\sigma(\mathbb{R}), u, v \in L^\sigma(\mathbb{R})$ we have

$$\begin{aligned} \|d_1(x)u\|_{L^\sigma} &\lesssim \|d_1(x)\|_{A^\sigma} \|u\|_{L^\sigma}, \\ \|d_2(x)v\|_{L^\sigma} &\lesssim \|d_2(x)\|_{A^\sigma} \|v\|_{L^\sigma}. \end{aligned}$$

For $0 < \delta \leq 1, \kappa' \leq 0, \kappa \geq 0$ we have

$$\begin{aligned} \|d_1(x)u\|_{Y_\delta^{\sigma, \kappa'}} &\lesssim \|d_1(x)\|_{A^\sigma(\mathbb{R})} \|u\|_{Y_\delta^{\sigma, \kappa}}, \\ \|d_2(x)v\|_{Y_{\alpha, \delta}^{\sigma, \kappa'}} &\lesssim \|d_2(x)\|_{A^\sigma(\mathbb{R})} \|v\|_{Y_\delta^{\sigma, \kappa}}. \end{aligned}$$

3. Bilinear Estimate

We state the desired bilinear estimate in the next lemma.

Lemma 3.1. (Bilinear estimate) *Let $\sigma > 0$ and $\kappa > \frac{1}{2}$. Then*

$$\|u \partial_x u\|_{Y^{\sigma, \kappa-1}} \lesssim_\kappa \|u\|_{Y^{\sigma, \kappa}}^2, \quad (3.1)$$

$$\|v \partial_x v\|_{Y^{\sigma, \kappa-1}} \lesssim_\kappa \|v\|_{Y_\alpha^{\sigma, \kappa}}^2, \quad (3.2)$$

$$\|u \partial_x u\|_{Y_\alpha^{\sigma, \kappa-1}} \lesssim_\kappa \|u\|_{Y^{\sigma, \kappa}}^2, \quad (3.3)$$

$$\|v \partial_x v\|_{Y_\alpha^{\sigma, \kappa-1}} \lesssim_\kappa \|v\|_{Y_\alpha^{\sigma, \kappa}}^2, \quad (3.4)$$

$$\|u \partial_x v\|_{Y^{\sigma, \kappa-1}} \lesssim_{\kappa} \|u\|_{Y^{\sigma, \kappa}} \|v\|_{Y_{\alpha}^{\sigma, \kappa}}, \quad (3.5)$$

$$\|v \partial_x u\|_{Y^{\sigma, \kappa-1}} \lesssim_{\kappa} \|u\|_{Y^{\sigma, \kappa}} \|v\|_{Y_{\alpha}^{\sigma, \kappa}}, \quad (3.6)$$

$$\|u \partial_x v\|_{Y_{\alpha}^{\sigma, \kappa-1}} \lesssim_{\kappa} \|u\|_{Y^{\sigma, \kappa}} \|v\|_{Y_{\alpha}^{\sigma, \kappa}}, \quad (3.7)$$

$$\|v \partial_x u\|_{Y_{\alpha}^{\sigma, \kappa-1}} \lesssim_{\kappa} \|u\|_{Y^{\sigma, \kappa}} \|v\|_{Y_{\alpha}^{\sigma, \kappa}}. \quad (3.8)$$

Proof. Let $\rho = \rho_1 + \rho_2$, we have $|\rho| \leq |\rho_1| + |\rho_2|$ by the triangle inequality. As a consequence, we then have

$$\cosh(\sigma|\rho|)(\operatorname{sech}(\sigma|\rho_1|)\operatorname{sech}(\sigma|\rho_2|)) = 2 \frac{(e^{\sigma|\rho|} + e^{-\sigma|\rho|})}{(e^{\sigma|\rho_1|} + e^{-\sigma|\rho_1|})(e^{\sigma|\rho_2|} + e^{-\sigma|\rho_2|})} \leq 4,$$

such that

$$\frac{(e^{\sigma|\rho|} + e^{-\sigma|\rho|})}{(e^{\sigma|\rho_1|} + e^{-\sigma|\rho_1|})(e^{\sigma|\rho_2|} + e^{-\sigma|\rho_2|})} \leq 2.$$

Setting

$$(U, V) = (\cosh(\sigma|\Lambda|)u, \cosh(\sigma|\Lambda|)v),$$

where (u, v) is the solution of (1.2). Then,

$$(u, v) = (\operatorname{sech}(\sigma|\Lambda|)U, \operatorname{sech}(\sigma|\Lambda|)V).$$

We start by proving the inequality (3.2).

$$\begin{aligned} \|\partial_x v^2\|_{Y^{\sigma, \kappa-1}} &= \|\cosh(\sigma|\Lambda|)[\partial_x (\operatorname{sech}(\sigma|\Lambda|)V)^2]\|_{Y^{\kappa-1}} \\ &= \|\rho \langle \tau - \rho^3 \rangle^{\kappa-1} \cosh(\sigma|\rho|) F_{x,t}[(\operatorname{sech}(\sigma|\Lambda|)V)^2]\|_{L_{\tau}^2 L_{\rho}^2} \\ &\lesssim \left\| \frac{|\rho|}{\langle \tau - \rho^3 \rangle^{1-\kappa}} \int_{\rho=\rho_1+\rho_2, \tau=\tau_1+\tau_2} |\tilde{V}(\rho_1, \tau_1)| |\tilde{V}(\rho_2, \tau_2)| d\rho_1 d\tau_1 \right\|_{L_{\tau}^2 L_{\rho}^2}, \\ &\lesssim \|\langle \tau - \rho^3 \rangle^{\kappa-1} |\rho| F_{x,t}(w^2)\|_{L_{\tau}^2 L_{\rho}^2}, \quad \text{where } w = F_{x,t}^{-1}(|\tilde{V}|), \\ &\sim \|\Lambda|(w^2)\|_{Y^{\kappa-1}} \\ &\lesssim_{\kappa} \|w\|_{Y_{\alpha}^{\kappa}}^2, \quad \text{by Proposition 5.3 in [9]} \\ &\sim \|V\|_{Y_{\alpha}^{\kappa}}^2 \sim \|v\|_{Y_{\alpha}^{\sigma, \kappa}}^2. \end{aligned}$$

Now, proving the inequality (3.7).

$$\begin{aligned} \|u \partial_x v\|_{Y_{\alpha}^{\sigma, \kappa-1}} &= \|\cosh(\sigma|\Lambda|)[(\operatorname{sech}(\sigma|\Lambda|)U) \partial_x (\operatorname{sech}(\sigma|\Lambda|)V)]\|_{Y_{\alpha}^{\kappa-1}} \\ &\lesssim \|\rho \langle \tau - \alpha \rho^3 \rangle^{\kappa-1} \cosh(\sigma|\rho|) F_{x,t}[(\operatorname{sech}(\sigma|\Lambda|)U)(\operatorname{sech}(\sigma|\Lambda|)V)]\|_{L_{\tau}^2 L_{\rho}^2} \\ &\lesssim \left\| \frac{|\rho|}{\langle \tau - \alpha \rho^3 \rangle^{1-\kappa}} \int_{\rho=\rho_1+\rho_2, \tau=\tau_1+\tau_2} |\tilde{U}(\rho_1, \tau_1)| |\tilde{V}(\rho_2, \tau_2)| d\rho_1 d\tau_1 \right\|_{L_{\tau}^2 L_{\rho}^2}, \\ &\lesssim \|\langle \tau - \alpha \rho^3 \rangle^{\kappa-1} |\rho| F_{x,t}(kw)\|_{L_{\tau}^2 L_{\rho}^2}, \quad \text{where } k = F_{x,t}^{-1}(|\tilde{U}|), w = F_{x,t}^{-1}(|\tilde{V}|) \\ &\sim \|\Lambda|(kw)\|_{Y_{\alpha}^{\kappa-1}} \\ &\lesssim_{\kappa} \|k\|_{Y^{\kappa}} \|w\|_{Y_{\alpha}^{\kappa}}, \quad \text{by Proposition 5.3 in [24]} \\ &\sim \|U\|_{Y^{\kappa}} \|V\|_{Y_{\alpha}^{\kappa}} \sim \|u\|_{Y^{\sigma, \kappa}} \|v\|_{Y_{\alpha}^{\sigma, \kappa}}. \end{aligned}$$

Lemma 3.2. (Energy-bilinear) Given $\kappa = \frac{1}{2} + \varepsilon$. We have $\forall \delta > 0, u \in Y_{\sigma, \kappa}(\mathbb{R}^2)$

$$\|\chi_{[0, \delta]}(\cdot) \Delta_1\|_{Y^{\kappa-1}} \lesssim \sigma^2 \|u\|_{Y^{\sigma, \kappa}}^2, \quad (3.9)$$

$$\|\chi_{[0, \delta]}(\cdot) \Delta_2\|_{Y^{\kappa-1}} \lesssim \sigma^2 \|v\|_{Y_{\alpha}^{\sigma, \kappa}}^2, \quad (3.10)$$

$$\|\chi_{[0, \delta]}(\cdot) \Delta_1\|_{Y_{\alpha}^{\kappa-1}} \lesssim \sigma^2 \|u\|_{Y^{\sigma, \kappa}}^2, \quad (3.11)$$

$$\|\chi_{[0, \delta]}(\cdot) \Delta_2\|_{Y_{\alpha}^{\kappa-1}} \lesssim \sigma^2 \|v\|_{Y_{\alpha}^{\sigma, \kappa}}^2, \quad (3.12)$$

$$\|\chi_{[0, \delta]}(\cdot) \Theta_1\|_{Y^{\kappa-1}} \lesssim \sigma^2 \|u\|_{Y^{\sigma, \kappa}} \|v\|_{Y_{\alpha}^{\sigma, \kappa}}, \quad (3.13)$$

$$\|\chi_{[0, \delta]}(\cdot) \Theta_2\|_{Y^{\kappa-1}} \lesssim \sigma^2 \|u\|_{Y^{\sigma, \kappa}} \|v\|_{Y_{\alpha}^{\sigma, \kappa}}, \quad (3.14)$$

$$\|\chi_{[0, \delta]}(\cdot) \Theta_1\|_{Y_{\alpha}^{\kappa-1}} \lesssim \sigma^2 \|u\|_{Y^{\sigma, \kappa}} \|v\|_{Y_{\alpha}^{\sigma, \kappa}}, \quad (3.15)$$

$$\|\chi_{[0, \delta]}(\cdot) \Theta_2\|_{Y_{\alpha}^{\kappa-1}} \lesssim \sigma^2 \|u\|_{Y^{\sigma, \kappa}} \|v\|_{Y_{\alpha}^{\sigma, \kappa}}, \quad (3.16)$$

Proof. By symmetry, we may assume that

$$|\rho_1| \geq |\rho_2| \geq \dots \geq |\rho_p|.$$

By Lemma [4],

$$\begin{aligned} |1 - \cosh(\sigma|\rho|) \prod_{j=1}^p \operatorname{sech}(\sigma|\rho_j|)| &\leq 2^p \sum_{j \neq k=1}^p |\sigma\rho_j| |\sigma\rho_k| \\ &\leq p^2 2^p \sigma^2 |\rho_1| |\rho_2|. \end{aligned} \quad (3.17)$$

We will prove (3.10) and (3.13), the rest is similar. Taking the Fourier transform, we have

$$F_x(\Delta_2)(\rho) = \frac{1}{2} \int_{\rho=\rho_1+\rho_2} |\rho| [1 - \cosh(\sigma|\rho|) \prod_{j=1}^2 \operatorname{sech}(\sigma|\rho_j|)] \prod_{j=1}^2 \widehat{V}^{\sigma}(\rho_j) d\rho_1 d\rho_2,$$

and

$$F_x(\Theta_1)(\rho) = \int_{\rho=\rho_1+\rho_2} [1 - \cosh(\sigma|\rho|) \prod_{j=1}^2 \operatorname{sech}(\sigma|\rho_j|)] \widehat{U}^{\sigma}(\rho_1) |\rho| \widehat{V}^{\sigma}(\rho_2) d\rho_1 d\rho_2.$$

Now set

$$\phi_1^{\sigma} = F_x^{-1}(|\widehat{U}^{\sigma}|),$$

and

$$\phi_2^{\sigma} = F_x^{-1}(|\widehat{V}^{\sigma}|).$$

Then applying (3.17)

$$\begin{aligned} |F_x(\Delta_1)(\rho)| &\leq 16\sigma^2 \int_{\rho=\rho_1+\rho_2} |\rho_1| |\rho_2| |\widehat{V}^{\sigma}(\rho_1)| |\widehat{V}^{\sigma}(\rho_2)| d\rho_1 d\rho_2 \\ &= 16\sigma^2 \int_{\rho=\rho_1+\rho_2} |\rho_1| |\rho_2| |\widehat{\phi}_2^{\sigma}(\rho_1)| |\widehat{\phi}_2^{\sigma}(\rho_2)| d\rho_1 d\rho_2 \\ &= 16\sigma^2 F_x((|\Lambda| \phi_2^{\sigma})^2)(\rho), \end{aligned}$$

and

$$\begin{aligned} |F_x(\Theta_1)(\rho)| &\leq 16\sigma^2 \int_{\rho=\rho_1+\rho_2} |\rho_1| |\rho_2| |\widehat{U}^{\sigma}(\rho_1)| |\widehat{V}^{\sigma}(\rho_2)| d\rho_1 d\rho_2 \\ &= 16\sigma^2 \int_{\rho=\rho_1+\rho_2} |\rho_1| |\rho_2| |\widehat{\phi}_1^{\sigma}(\rho_1)| |\widehat{\phi}_2^{\sigma}(\rho_2)| d\rho_1 d\rho_2 \\ &= 16\sigma^2 F_x((|\Lambda| \phi_1^{\sigma} |\Lambda| \phi_2^{\sigma}))(\rho). \end{aligned}$$

Thanks to the Plancherel, Hölder and Sobolev embedding, we get

$$\begin{aligned} |\chi_{[0, \delta]}(\cdot) \Delta_1| &\lesssim \sigma^2 \|(|\Lambda| \phi_2^{\sigma})^2\|_{Y^{\kappa-1}} \\ &\lesssim \sigma^2 \|\phi_2^{\sigma}\|_{Y_{\alpha}^{\kappa}}^2 \\ &\sim \sigma^2 \|V^{\sigma}\|_{Y_{\alpha}^{\kappa}}^2, \end{aligned}$$

and

$$\begin{aligned} |\kappa_{[0,\delta]}(\cdot)\Theta_1| &\lesssim \sigma^2 \|(|\Lambda|\phi_1^\sigma |\Lambda|\phi_2^\sigma)\|_{Y_\alpha^{\kappa-1}} \\ &\lesssim \sigma^2 \|\phi_2^\sigma\|_{Y_\alpha^\kappa} \|\phi_1^\sigma\|_{Y^\kappa} \\ &\sim \sigma^2 \|V^\sigma\|_{Y_\alpha^\kappa} \|U^\sigma\|_{Y^\kappa}. \end{aligned}$$

Lemma 3.3. [11] Let $d_1, d_2 \in A^{\sigma_0}$ where $\sigma_0 > 0$. Then we have $\forall 0 < \sigma \leq \sigma_0$

$$\begin{aligned} \|\Gamma_1\|_{L^2} &\leq 2 \frac{\sigma}{\sigma_0} \|d_1\|_{A^{\sigma_0}} \|u\|_{L^\sigma} + 2 \|d_1\|_{L^\infty} \|u\|_{L^2}, \\ \|\Gamma_2\|_{L^2} &\leq 2 \frac{\sigma}{\sigma_0} \|d_2\|_{A^{\sigma_0}} \|u\|_{L^\sigma} + 2 \|d_2\|_{L^\infty} \|u\|_{L^2}. \end{aligned}$$

4. Local Well Posedness

We begin with our first main result: a local well-posedness theorem. We denote $B = \max\{\beta_i, i = 1, \dots, 8\}$.

Theorem 4.1. Let $\sigma > 0$ and $(u_0, v_0) \in L^\sigma$. Then there exists $\kappa > \frac{1}{2}$ and $\delta > 0$ depending only on σ, s and u_0, v_0 , where (1.2) is locally well-posed in

$$C([0, \delta], L^\sigma(\mathbb{R})) \times C([0, \delta], L^\sigma(\mathbb{R})).$$

In addition, the solution (u, v) satisfy

$$\|(u(t), v(t))\|_{\mathbb{L}^\sigma} \leq c \|(u, v)\|_{Y_\alpha^{\sigma, \kappa}} \leq 2C \|(u_0, v_0)\|_{\mathbb{L}^\sigma}, \quad t \in [0, \delta].$$

Proof. Let $B(0, r)$ be a closed ball given by

$$B(0, r) = \{(u, v) \in \|(u, v)\|_{Y_\alpha^{\sigma, \kappa}} \leq r\} \text{ here } r = 2C \|(u_0, v_0)\|_{\mathbb{L}^\sigma}.$$

It will be shown that the map

$$(\Phi(u), \Psi(v)): B(0, r) \rightarrow B(0, r),$$

is a contraction. By using Lemma 2.1, Lemma 2.4 and Lemma 3.1, we get

$$\|(\Phi_1(u), \Psi_1(v))\|_{Y_\alpha^{\sigma, \kappa}} \lesssim r,$$

and

$$\|(\Phi(u) - \Phi(u^*), \Psi(v) - \Psi(v^*))\|_{Y_\alpha^{\sigma, \kappa}} \lesssim \frac{1}{2} \|(u - u^*), (v - v^*)\|_{Y_\alpha^{\sigma, \kappa}},$$

where

$$\delta: \frac{1}{(1 + \|(d_1, d_2)\|_{A^{\sigma_0}} + 4\|(u_0, v_0)\|_{\mathbb{L}^\sigma})^{\frac{1}{\kappa^{\frac{1}{2}} - \kappa}}}.$$

5. Lower Bound of Analytic Radius

In our second main result, for a system of modified KdV equations with damping, the analytic radius has a fixed positive lower bound uniformly for all time.

Theorem 5.1. We suppose that the damping term assumptions (A1) and (A2) hold and (u_0, v_0) belongs to $\mathbb{L}^{\sigma_0}$, for some $\sigma_0 > 0$. Then, system (1.2) has a unique global solution $(u(t), v(t))$ belongs to $\mathbb{L}^{\sigma(t)}$, where

$$\sigma(t) \geq \tilde{\sigma}_0, \quad \forall t \geq 0,$$

with $\tilde{\sigma}_0 > 0$, depends only on $\|(u_0, v_0)\|_{\mathbb{L}^{\sigma_0}}$, σ_0 , d_1 and d_2 . Furthermore

$$\|(u(t), v(t))\|_{\mathbb{L}^{\tilde{\sigma}_0}} \leq C e^{\frac{-\lambda t}{2}}, \quad \forall t \geq 0,$$

and $C > 0$ depends only on $\|(u_0, v_0)\|_{\mathbb{L}^{\sigma_0}}$, d_1 and d_2 .

In this subsection, we derive global bounds for the local solution obtained in Theorem 4.1. Following the strategy in [4, 8, 11], the argument proceeds in two steps:

5.1. Almost Conservation Law

Recalling that

$$\frac{d}{dt} \int_{\mathbb{R}} u^2 dx + 2 \int_{\mathbb{R}} d_1(x) u^2 dx = 0, \quad \forall t \geq 0, \quad (5.1)$$

$$\frac{d}{dt} \int_{\mathbb{R}} v^2 dx + 2 \int_{\mathbb{R}} d_2(x) v^2 dx = 0, \quad \forall t \geq 0. \quad (5.2)$$

If $d_i(x) \geq T > 0, i = 1, 2$ on $\mathbb{R}$, so by the Gronwall's inequality, we find by (5.1) and (5.2) that

$$\|(u(t), v(t))\|_{\mathbb{L}^2}^2 \leq e^{-2Tt} \|(u_0, v_0)\|_{\mathbb{L}^2}^2, \quad \forall t \geq 0. \quad (5.3)$$

The main goal here is to establish an energy growth bound in the analytic function spaces L^σ . The statement of the result is as follows.

Theorem 5.2. Assume that $(d_1, d_2) \in \mathbb{A}^{\sigma_0}$ with some $\sigma_0 > 0$ and $d_1(x), d_2(x) \geq T > 0$ for all $x \in \mathbb{R}$. Let $0 < \sigma < \sigma_0$ and $(u_0, v_0) \in \mathbb{L}^\sigma$, and let (u, v) be the solution on $[0, \delta]$ obtained in Theorem 4.1. Then there exist constants $C_1, C_2, C_3 > 0$ depending only on T and σ_0 so that

$$\begin{aligned} \|(u(\delta), v(\delta))\|_{\mathbb{L}^\sigma}^2 &\lesssim e^{-2T\delta} \|(u_0, v_0)\|_{\mathbb{L}^\sigma}^2 + C_1 \sigma^2 \|(u_0, v_0)\|_{\mathbb{L}^\sigma}^3 \\ &+ C_2 \sigma \|(d_1, d_2)\|_{\mathbb{A}^{\sigma_0}} \|(u_0, v_0)\|_{\mathbb{L}^\sigma}^2 + C_3 \|(d_1, d_2)\|_{\mathbb{L}^\infty} \|(u_0, v_0)\|_{\mathbb{L}^2(\mathbb{R})} \|(u_0, v_0)\|_{\mathbb{L}^\sigma}. \end{aligned}$$

Proof. By (1.2), it is not hard to see

$$\begin{cases} \partial_t(\cosh(\sigma|\Lambda|)u) + \partial_x^3(\cosh(\sigma|\Lambda|)u) + \beta_1 \cosh(\sigma|\Lambda|)(u \partial_x u) + \beta_2 \cosh(\sigma|\Lambda|)(u \partial_x v) + \beta_3 \cosh(\sigma|\Lambda|)(v \partial_x u) \\ + \beta_4 \cosh(\sigma|\Lambda|)(v \partial_x v) + \cosh(\sigma|\Lambda|)(d_1(x)u) = 0, \\ \partial_t(\cosh(\sigma|\Lambda|)v) + \alpha \partial_x^3(\cosh(\sigma|\Lambda|)v) + \beta_5 \cosh(\sigma|\Lambda|)(u \partial_x u) + \beta_6 \cosh(\sigma|\Lambda|)(u \partial_x v) + \beta_8 \cosh(\sigma|\Lambda|)(v \partial_x u) \\ + \beta_8 \cosh(\sigma|\Lambda|)(v \partial_x v) + \cosh(\sigma|\Lambda|)(d_2(x)v) = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} \partial_t(\cosh(\sigma|\Lambda|)u) + \partial_x^3(\cosh(\sigma|\Lambda|)u) + \beta_1(\cosh(\sigma|\Lambda|)u \partial_x \cosh(\sigma|\Lambda|)u) + \beta_2(\cosh(\sigma|\Lambda|)u \partial_x \cosh(\sigma|\Lambda|)v) \\ + \beta_3(\cosh(\sigma|\Lambda|)v \partial_x \cosh(\sigma|\Lambda|)u) + \beta_4(\cosh(\sigma|\Lambda|)v \partial_x \cosh(\sigma|\Lambda|)v) + (d_1(x) \cosh(\sigma|\Lambda|)u) \\ = \beta_1(\cosh(\sigma|\Lambda|)u \partial_x \cosh(\sigma|\Lambda|)u) + \beta_2(\cosh(\sigma|\Lambda|)u \partial_x \cosh(\sigma|\Lambda|)v) + \beta_3(\cosh(\sigma|\Lambda|)v \partial_x \cosh(\sigma|\Lambda|)u) \\ + \beta_4(\cosh(\sigma|\Lambda|)v \partial_x \cosh(\sigma|\Lambda|)v) + (d_1(x) \cosh(\sigma|\Lambda|)u) - \beta_1 \cosh(\sigma|\Lambda|)(u \partial_x u) - \beta_2 \cosh(\sigma|\Lambda|)(u \partial_x v) \\ - \beta_3 \cosh(\sigma|\Lambda|)(v \partial_x u) - \beta_4 \cosh(\sigma|\Lambda|)(v \partial_x v) - \cosh(\sigma|\Lambda|)(d_1(x)u), \\ \partial_t(\cosh(\sigma|\Lambda|)v) + \alpha \partial_x^3(\cosh(\sigma|\Lambda|)v) + \beta_5(\cosh(\sigma|\Lambda|)u \partial_x \cosh(\sigma|\Lambda|)u) + \beta_6(\cosh(\sigma|\Lambda|)u \partial_x \cosh(\sigma|\Lambda|)v) \\ + \beta_7(\cosh(\sigma|\Lambda|)v \partial_x \cosh(\sigma|\Lambda|)u) + \beta_8(\cosh(\sigma|\Lambda|)v \partial_x \cosh(\sigma|\Lambda|)v) + (d_2(x) \cosh(\sigma|\Lambda|)u) \\ = \beta_5(\cosh(\sigma|\Lambda|)u \partial_x \cosh(\sigma|\Lambda|)u) + \beta_6(\cosh(\sigma|\Lambda|)u \partial_x \cosh(\sigma|\Lambda|)v) + \beta_7(\cosh(\sigma|\Lambda|)v \partial_x \cosh(\sigma|\Lambda|)u) \\ + \beta_8(\cosh(\sigma|\Lambda|)v \partial_x \cosh(\sigma|\Lambda|)v) + (d_2(x) \cosh(\sigma|\Lambda|)u) - \beta_5 \cosh(\sigma|\Lambda|)(u \partial_x u) - \beta_6 \cosh(\sigma|\Lambda|)(u \partial_x v) \\ - \beta_7 \cosh(\sigma|\Lambda|)(v \partial_x u) - \beta_8 \cosh(\sigma|\Lambda|)(v \partial_x v) - \cosh(\sigma|\Lambda|)(d_2(x)u), \end{cases}$$

We use $\Delta_1, \Delta_2, \Theta_1, \Theta_2, \Gamma_1$ and Γ_2 , we obtain

$$\begin{cases}
\partial_t(\cosh(\sigma|\Lambda|)u) + \partial_x^3(\cosh(\sigma|\Lambda|)u) + \beta_1(\cosh(\sigma|\Lambda|)u \partial_x \cosh(\sigma|\Lambda|)u) + \beta_2(\cosh(\sigma|\Lambda|)u \partial_x \cosh(\sigma|\Lambda|)v) \\
+ \beta_3(\cosh(\sigma|\Lambda|)v \partial_x \cosh(\sigma|\Lambda|)u) + \beta_4(\cosh(\sigma|\Lambda|)v \partial_x \cosh(\sigma|\Lambda|)v) + (d_1(x) \cosh(\sigma|\Lambda|)u) \\
= \beta_1\Delta_1 + \beta_2\Theta_1 + \beta_3\Theta_2 + \beta_4\Delta_2 + \Gamma_1 \\
\partial_t(\cosh(\sigma|\Lambda|)v) + \alpha \partial_x^3(\cosh(\sigma|\Lambda|)v) + \beta_5(\cosh(\sigma|\Lambda|)u \partial_x \cosh(\sigma|\Lambda|)u) + \beta_6(\cosh(\sigma|\Lambda|)u \partial_x \cosh(\sigma|\Lambda|)v) \\
+ \beta_7(\cosh(\sigma|\Lambda|)v \partial_x \cosh(\sigma|\Lambda|)u) + \beta_8(\cosh(\sigma|\Lambda|)v \partial_x \cosh(\sigma|\Lambda|)v) + (d_2(x) \cosh(\sigma|\Lambda|)u) \\
= \beta_5\Delta_1 + \beta_6\Theta_1 + \beta_7\Theta_2 + \beta_8\Delta_2 + \Gamma_2
\end{cases} \quad (5.4)$$

Multiplying (5.4)₁ by $\cosh(\sigma|\Lambda|)u$ and (5.4)₂ by $\cosh(\sigma|\Lambda|)v$ and integrating in space and by assuming that $\cosh(\sigma|\Lambda|)u$ and $\cosh(\sigma|\Lambda|)v$ decay to zero when $|x| \rightarrow \infty$, and the same applies to all spatial derivatives. Then, (5.4) becomes

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)u)^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)v)^2 dx + \int_{\mathbb{R}} d_1(x) (\cosh(\sigma|\Lambda|)u)^2 dx + \int_{\mathbb{R}} d_2(x) (\cosh(\sigma|\Lambda|)v)^2 dx \\
& = \beta_1 \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)u) \Delta_1 dx + \beta_4 \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)u) \Delta_2 dx + \beta_2 \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)u) \Theta_1 dx + \beta_3 \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)u) \Theta_2 dx \\
& + \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)u) \Gamma_1 dx + \beta_5 \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)v) \Delta_1 dx + \beta_8 \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)v) \Delta_2 dx + \beta_6 \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)v) \Theta_1 dx \\
& + \beta_7 \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)v) \Theta_2 dx + \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)v) \Gamma_2 dx
\end{aligned}$$

Since $d_1(x), d_2(x) \geq T > 0, \forall x \in \mathbb{R}$ we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)u)^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)v)^2 dx + T \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)u)^2 dx + T \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)v)^2 dx \\
& \lesssim_B \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)u) \Delta_1 dx + \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)u) \Delta_2 dx + \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)u) \Theta_1 dx + \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)u) \Theta_2 dx \\
& + \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)u) \Gamma_1 dx + \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)v) \Delta_1 dx + \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)v) \Delta_2 dx + \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)v) \Theta_1 dx \\
& + \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)v) \Theta_2 dx + \int_{\mathbb{R}} (\cosh(\sigma|\Lambda|)v) \Gamma_2 dx
\end{aligned}$$

Applying Gronwall's lemma, we have

$$\begin{aligned}
\|u(\delta), v(\delta)\|_{\mathbb{L}^\sigma}^2 & \lesssim_B e^{-2T\delta} \|(u(0), v(0))\|_{\mathbb{L}^\sigma}^2 \\
& + 2 \left| \int_{\mathbb{R}^2} \chi_{[0, \delta]}(t) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)u) \Delta_1 dx d\tau \right| \\
& + 2 \left| \int_{\mathbb{R}^2} \chi_{[0, \delta]}(t) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)u) \Delta_2 dx d\tau \right| \\
& + 2 \left| \int_{\mathbb{R}^2} \chi_{[0, \delta]}(t) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)v) \Delta_1 dx d\tau \right| \\
& + 2 \left| \int_{\mathbb{R}^2} \chi_{[0, \delta]}(t) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)v) \Delta_2 dx d\tau \right| \\
& + 2 \left| \int_{\mathbb{R}^2} \chi_{[0, \delta]}(t) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)u) \Theta_1 dx d\tau \right| \\
& + 2 \left| \int_{\mathbb{R}^2} \chi_{[0, \delta]}(t) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)u) \Theta_2 dx d\tau \right| \\
& + 2 \left| \int_{\mathbb{R}^2} \chi_{[0, \delta]}(t) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)v) \Theta_1 dx d\tau \right| \\
& + 2 \left| \int_{\mathbb{R}^2} \chi_{[0, \delta]}(t) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)v) \Theta_2 dx d\tau \right| \\
& + 2 \left| \int_0^T \int_{\mathbb{R}} \chi_{[0, \delta]}(t) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)u) \Gamma_1 dx d\tau \right| \\
& + 2 \left| \int_0^T \int_{\mathbb{R}} \chi_{[0, \delta]}(t) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)v) \Gamma_2 dx d\tau \right| \\
& \leq e^{-2T\delta} \|(u(0), v(0))\|_{\mathbb{L}^\sigma}^2 + I_1 + I_2 + I_3 + I_4 + I_5 \\
& + I_6 + I_7 + I_8 + I_9 + I_{10}.
\end{aligned}$$

Estimate I_1 : Since $(\cosh(\sigma|\Lambda|)u)$, $(\cosh(\sigma|\Lambda|)v)$, are real-valued, it follows from Plancherel's identity and Cauchy-Schwarz inequality that

$$\begin{aligned} I_1 &= 2 \left| \int_{\mathbb{R}^2} (\chi_{[0,\delta]}(\cdot) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)u))(\rho, \tau) \overline{(\chi_{[0,\delta]}(\cdot) \Delta_1)}(\rho, \tau) d\rho d\tau \right| \\ &= 2 \left| \int_{\mathbb{R}^2} (\chi_{[0,\delta]}(\cdot) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)u))(\rho, \tau) \overline{(\chi_{[0,\delta]}(\cdot) \Delta_1)}(\rho, \tau) d\rho d\tau \right| \\ &\leq 2 \left\| \chi_{[0,\delta]}(\cdot) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)u) \right\|_{Y^{1-\kappa}} \left\| \chi_{[0,\delta]}(\cdot) \Delta_1 \right\|_{Y^{\kappa-1}}. \end{aligned}$$

Estimate I_2 : We have

$$I_2 \leq 2 \left\| \chi_{[0,\delta]}(\cdot) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)u) \right\|_{Y^{1-\kappa}} \left\| \chi_{[0,\delta]}(\cdot) \Delta_2 \right\|_{Y^{\kappa-1}}, \quad (5.5)$$

Estimate I_3 : We have

$$I_3 \leq 2 \left\| \chi_{[0,\delta]}(\cdot) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)v) \right\|_{Y_\alpha^{1-\kappa}} \left\| \chi_{[0,\delta]}(\cdot) \Delta_1 \right\|_{Y_\alpha^{\kappa-1}}, \quad (5.6)$$

Estimate I_4 : We have

$$I_4 \leq 2 \left\| \chi_{[0,\delta]}(\cdot) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)v) \right\|_{Y_\alpha^{1-\kappa}} \left\| \chi_{[0,\delta]}(\cdot) \Delta_2 \right\|_{Y_\alpha^{\kappa-1}}, \quad (5.7)$$

Estimate I_5 : We have

$$I_5 \leq 2 \left\| \chi_{[0,\delta]}(\cdot) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)u) \right\|_{Y^{1-\kappa}} \left\| \chi_{[0,\delta]}(\cdot) \Theta_1 \right\|_{Y^{\kappa-1}}, \quad (5.8)$$

Estimate I_6 : We have

$$I_6 \leq 2 \left\| \chi_{[0,\delta]}(\cdot) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)u) \right\|_{Y^{1-\kappa}} \left\| \chi_{[0,\delta]}(\cdot) \Theta_2 \right\|_{Y^{\kappa-1}}, \quad (5.9)$$

Estimate I_7 : We have

$$I_7 \leq 2 \left\| \chi_{[0,\delta]}(\cdot) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)v) \right\|_{Y_\alpha^{1-\kappa}} \left\| \chi_{[0,\delta]}(\cdot) \Theta_1 \right\|_{Y_\alpha^{\kappa-1}}, \quad (5.10)$$

Estimate I_8 : We have

$$I_8 \leq 2 \left\| \chi_{[0,\delta]}(\cdot) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)v) \right\|_{Y_\alpha^{1-\kappa}} \left\| \chi_{[0,\delta]}(\cdot) \Theta_2 \right\|_{Y_\alpha^{\kappa-1}}, \quad (5.11)$$

we have both $-\frac{1}{2} < \kappa - 1 < \frac{1}{2}$ and $\frac{1}{2} < 1 - \kappa < \frac{1}{2}$. Therefore, one can use Lemma 2.3, Lemma 2.2 to obtain

$$\begin{aligned} \left\| \chi_{[0,\delta]}(\cdot) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)u) \right\|_{Y^{1-\kappa}} &= \left\| \chi_{[0,\delta]}(\cdot) e^{-2T(\delta-\tau)} u \right\|_{Y^{\sigma, 1-\kappa}} \\ &\lesssim \|(u, v)\|_{Y_\alpha^{\sigma, 1-\kappa}}, \\ \left\| \chi_{[0,\delta]}(\cdot) e^{-2T(\delta-\tau)} (\cosh(\sigma|\Lambda|)v) \right\|_{Y^{1-\kappa}} &= \left\| \chi_{[0,\delta]}(\cdot) e^{-2T(\delta-\tau)} v \right\|_{Y_\alpha^{\sigma, 1-\kappa}} \\ &\lesssim \|(u, v)\|_{Y_\alpha^{\sigma, 1-\kappa}}. \end{aligned}$$

Then, we can conclude by Lemma 3.2 and the fact that $1 - \kappa < \kappa$

$$\sum_{i=1}^8 I_i \lesssim \sigma^2 \|(u, v)\|_{Y_\alpha^{\sigma, \kappa}}^3 \lesssim \sigma^2 \|(u_0, v_0)\|_{\mathbb{L}^\sigma}^3.$$

Estimate I_9 and I_{10} : By using Lemma 3.3, Lemma 2.3 and Lemma 2.4 we get

$$\sum_{i=9}^{10} I_i \leq C_2 \sigma \|(d_1, d_2)\|_{\mathbb{A}^{\sigma_0}} \|(u_0, v_0)\|_{\mathbb{L}^\sigma}^2 + C_3 \|(d_1, d_2)\|_{\mathbb{L}^\infty} \|(u_0, v_0)\|_{\mathbb{L}^2(\mathbb{R})} \|(u_0, v_0)\|_{\mathbb{L}^\sigma}.$$

5.2. Proof of Theorem 5.1

By assumption (A2), we get

$$\|d_1\|_{A^\sigma(\mathbb{R})} < \sum_{k \geq 0} (k+1)^{\frac{1}{4}} \frac{\sigma^k}{k!} \|\partial_x^k u\|_{L^\infty(\mathbb{R})} \leq \sum_{k \geq 0} (k+1)^{\frac{1}{4}} \sigma^k M_1 M_2^k < \infty, \quad \text{for all } 0 < \sigma < M_2^{-1}.$$

and

$$\|d_2\|_{A^\sigma(\mathbb{R})} < \sum_{k \geq 0} (k+1)^{\frac{1}{4}} \frac{\sigma^k}{k!} \|\partial_x^k v\|_{L^\infty(\mathbb{R})} \leq \sum_{k \geq 0} (k+1)^{\frac{1}{4}} \sigma^k M_1 M_2^k < \infty, \quad \text{for all } 0 < \sigma < M_2^{-1}.$$

Let $(u_0, v_0) \in \mathbb{L}^{\sigma_0}(\mathbb{R})$ where $\sigma_0 < M_2^{-1}$. Thus, we may prove that

$$(d_1(x), d_2(x)) \in \mathbb{A}^{\sigma_0}(\mathbb{R}).$$

By Theorem 4.1, it exists a unique solution

$$(u, v) \in C([0, \delta], L^\sigma(\mathbb{R})) \times C([0, \delta], L^\sigma(\mathbb{R})),$$

of (1.2) where

$$\delta \sim \frac{1}{(1 + \|(d_1, d_2)\|_{\mathbb{A}^{\sigma_0}} + 4\|(u_0, v_0)\|_{\mathbb{L}^{\sigma_0}})^{\frac{1}{\kappa' - \kappa}}}.$$

and by Theorem 5.2, we have

$$\begin{aligned} \|(u(\delta), v(\delta))\|_{\mathbb{L}^\sigma}^2 &\leq e^{-2T\delta} \|(u_0, v_0)\|_{\mathbb{L}^\sigma}^2 + C_1 \sigma^2 \|(u_0, v_0)\|_{\mathbb{L}^\sigma}^3 \\ &+ C_2 \sigma \|(d_1, d_2)\|_{\mathbb{A}^{\sigma_0}} \|(u_0, v_0)\|_{\mathbb{L}^\sigma}^2 + C_3 \|(d_1, d_2)\|_{\mathbb{L}^\infty} \|(u_0, v_0)\|_{\mathbb{L}^2(\mathbb{R})} \|(u_0, v_0)\|_{\mathbb{L}^\sigma}. \end{aligned}$$

Now, using the inequality

$$C_3 \|(d_1, d_2)\|_{\mathbb{L}^\infty} \|(u_0, v_0)\|_{\mathbb{L}^2(\mathbb{R})} \|(u_0, v_0)\|_{\mathbb{L}^\sigma} \leq \frac{1 - e^{-2T\delta}}{2} \|(u_0, v_0)\|_{\mathbb{L}^\sigma}^2 + C_4 \|(u_0, v_0)\|_{\mathbb{L}^2(\mathbb{R})}^2,$$

with $C_4 = (1 - e^{-2T\delta})^{-1} (C_3 \|(d_1, d_2)\|_{\mathbb{L}^\infty})^2$, we deduce that

$$\begin{aligned} \|(u(\delta), v(\delta))\|_{\mathbb{L}^\sigma}^2 &\leq \frac{1 + e^{-2T\delta}}{2} \|(u_0, v_0)\|_{\mathbb{L}^\sigma}^2 + C_1 \sigma^2 \|(u_0, v_0)\|_{\mathbb{L}^\sigma}^3 \\ &+ C_2 \sigma \|(d_1, d_2)\|_{\mathbb{A}^{\sigma_0}} \|(u_0, v_0)\|_{\mathbb{L}^\sigma}^2 + \frac{1 - e^{-2T\delta}}{2} \|(u_0, v_0)\|_{\mathbb{L}^\sigma}^2 + C_4 \|(u_0, v_0)\|_{\mathbb{L}^2(\mathbb{R})}^2. \end{aligned}$$

Now, we choose

$$\sigma = \tilde{\sigma}_0 = \min\left\{\sigma_0, \frac{1 - e^{-2T\delta}}{4C_2 \|(d_1, d_2)\|_{\mathbb{A}^{\sigma_0}}}, \left(\frac{1 - e^{-2T\delta}}{4C_1 \|(u_0, v_0)\|_{\mathbb{L}^\sigma}^2}\right)^{\frac{1}{2}}\right\}.$$

So

$$C_1 \sigma^2 \|(u_0, v_0)\|_{\mathbb{L}^\sigma}^3 + C_2 \sigma \|(d_1, d_2)\|_{\mathbb{A}^{\sigma_0}} \|(u_0, v_0)\|_{\mathbb{L}^\sigma}^2 \leq \frac{1 - e^{-2T\delta}}{2}.$$

Then

$$\|(u(\delta), v(\delta))\|_{\mathbb{L}^{\tilde{\sigma}_0}}^2 \leq \|(u_0, v_0)\|_{\mathbb{L}^{\tilde{\sigma}_0}}^2 + C_4 \|(u_0, v_0)\|_{\mathbb{L}^2}^2. \quad (5.12)$$

The solution on $[n\delta, (n+1)\delta]$ proceeds as above

$$\|(u(n\delta), v(n\delta))\|_{\mathbb{L}^{\tilde{\sigma}_0}}^2 \leq \|(u_0, v_0)\|_{\mathbb{L}^{\tilde{\sigma}_0}}^2 + C_4 \sum_{i=1}^n e^{-2(i-1)T\delta} \|(u_0, v_0)\|_{\mathbb{L}^2}^2, \quad \forall n \geq 1. \quad (5.13)$$

We obtain

$$\|(u(t), v(t))\|_{\mathbb{L}^{\tilde{\sigma}_0}}^2 \leq \|(u_0, v_0)\|_{\mathbb{L}^{\tilde{\sigma}_0}}^2 + C_5 \|(u_0, v_0)\|_{\mathbb{L}^2}^2, \quad \forall t \geq 0. \quad (5.14)$$

where $C_5 > 0$ is independent on n . By ((5.14)), we get

$$\|(u(t), v(t))\|_{\mathbb{L}^{\tilde{\sigma}_0}} \leq C_6, \quad \forall t \geq 0. \quad (5.15)$$

Recalling the exponential decay

$$\|(u(t), v(t))\|_{\mathbb{L}^2} \leq e^{-T\delta} \|(u_0, v_0)\|_{\mathbb{L}^2},$$

we deduce from ((5.15)) that

$$\|(u(t), v(t))\|_{\frac{\sigma_0}{2}} \leq \|(u(t), v(t))\|_{\frac{1}{2}}^{\frac{1}{2}} \|(u(t), v(t))\|_{\frac{1}{2}}^{\frac{1}{2}} \leq \sqrt{C_6} e^{\frac{-Tt}{2}} \|(u_0, v_0)\|_{\frac{1}{2}}^{\frac{1}{2}}, \quad \forall t \geq 0.$$

The proof of Theorem 5.1 is completed.

Conflict of Interest

The author declares that there is no conflict of interest regarding the publication of this editorial.

Funding

No specific funding was received for the writing or publication of this editorial.

Acknowledgement

The author would like to thank the contributors and reviewers of this issue for their valuable efforts and insights.

References

- [1] Atmani A, Boukarou A, Benterki D, Zennir K. Spatial analyticity of solutions for a coupled system of generalized KdV equations. *Math Meth Appl Sci*. 2024;47(12):10351–72. doi:10.48550/arXiv.2201.12864
- [2] Boukarou A, Zennir K, Bouye M, Moumen A. Nondecreasing analytic radius for the Kawahara–Korteweg–de Vries equation. *AIMS Math*. 2024;9(8):22414–34. doi:10.3934/math.20241090
- [3] Boukarou A, Zennir K, Guerbaty K, Georgiev SG. Well-posedness and regularity of the fifth order Kadomtsev–Petviashvili I equation in the analytic Bourgain spaces. *Ann Univ Ferrara Sez VII Sci Mat*. 2020;66:255–72. doi:10.1007/s11565-020-00340-8
- [4] Dufera T, Mebrate S, Tesfahun A. On the persistence of spatial analyticity for the beam equation. *J Math Anal Appl*. 2022;509(2):126001. doi:10.1016/j.jmaa.2022.126001
- [5] Faminskii AV, Nikolayev A. On stationary solutions of KdV and mKdV equations. *Differential and Difference Equations with Applications*. 2016;164:63–70.
- [6] Otmani S, Boukarou A, Zennir K, Bouhali K, Moumen A, Bouye M. On the study of the radius of analyticity for Korteweg–de Vries type systems with weak damping. *AIMS Math*. 2024;9(10):28341–60. doi:10.3934/math.20241375
- [7] Petronilho G, da Silva PL. On the radius of spatial analyticity for the modified Kawahara equation on the line. *Math Nachr*. 2019;292(9):2032–47. doi:10.1002/mana.201800394
- [8] Selberg S, Silva DO. Lower bounds on the radius of spatial analyticity for the KdV equation. *Ann Henri Poincaré*. 2016;18:1009–23. doi:10.1007/s00023-016-0498-1
- [9] Oh T. Diophantine conditions in well-posedness theory of coupled KdV-type systems: local theory. *Int Math Res Not*. 2009;18:3516–56. doi:10.1093/imrn/rnn083
- [10] Elmansouri A, Zennir K, Boukarou A, Zehrou O. Analytic Gevrey well-posedness and regularity for class of coupled periodic KdV systems of Majda–Biello type. *Appl Sci*. 2022;12:117–30.
- [11] Wang M. Nondecreasing analytic radius for the KdV equation with weak damping. *Nonlinear Anal*. 2022;215:112653. doi:10.1016/j.na.2022.112653