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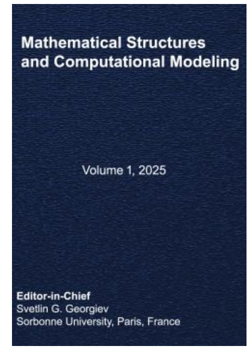

A New Introduction to Riemann-Liouville Fractional Sobolev Spaces within Time Scale Frameworks

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ABSTRACT

We present a new concept of Riemann-Liouville Fractional Sobolev Spaces in the context of time scale calculus in this paper. This novel method unifies continuous and discrete analysis by extending conventional fractional Sobolev space theory to dynamic domains. Our study's primary contribution is the first description of L_{Δ}^{p}-representability in relation to time scales, which lays the groundwork for future advancements in fractional dynamic equations. Our findings offer novel ideas regarding fractional.

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1. Introduction and Basic Concepts

One of the most active and developing areas of contemporary mathematical investigations in the 17th century, and Riemann, Liouville, and Caputo's seminal work subsequently formalized it. Fractional derivatives have proven highly effective in modeling complex physical and biological systems characterized by memory and hereditary behaviors, such as anomalous diffusion and viscoelasticity.

The theory of time scales was first presented by Stefan Hilger [4] in 1988 with the goal of combining the analysis of discrete and continuous dynamic systems into a single analytical framework. A new discipline called dynamic equations on time scales has emerged as a result of this theory, which has made it possible to comprehend hybrid dynamic behaviors on a deeper level.

Researchers like Bohner, Peterson [11, 12], and Atici have been working to include fractional calculus into the time scale framework since the early 2000s. Benchohra and Goodrich have since contributed to these efforts. New models and definitions for fractional dynamic equations on time scales were presented in these investigations.

One major breakthrough was the generalization of Lebesgue integration on time scales, as detailed in the work of Cabada *et al.* [1], who demonstrated how the Δ -integral can be expressed as a usual Lebesgue integral.

Building upon this, the construction of Lebesgue spaces $L^p_\Delta(T,R)$ was established as a necessary step for analyzing Δ -measurable functions. In this context, Benaissa *et al.* [2] investigated density problems in $L^p_\Delta(T,R)$ spaces, laying the groundwork for further functional analysis on time scales.

As a natural extension, Sobolev spaces on time scales were studied to incorporate weak differentiability and variational structures. The approximation and density properties of such spaces were examined in depth by Ladrani *et al.* [5, 6] in their article Density Problems in Sobolev's Spaces on Time Scales, where they provided key insights into the behavior of Sobolev-type embeddings. These cumulative contributions significantly advance the foundation for extending fractional Sobolev spaces on time scales, motivating the development presented in this current study.

In this paper, we introduce a new definition of Riemann–Liouville fractional Sobolev spaces on time scales. We investigate their main properties and topological structure, focusing on how these spaces behave under the framework of fractional time-scale calculus. Special attention is given to their relationship with classical functional spaces defined on time scales. This study aims to extend the functional-analytic tools available for fractional dynamic equations in hybrid temporal settings.

1.1. Fundamental Notions in Time Scale Calculus

A time scale T is an arbitrary nonempty closed subset of the real numbers R. The jump operators $\sigma,\rho: T \to R$ are defined by:

$$\begin{cases} \sigma(t) = inf(t,\infty) \cap T \\ \rho(t) = inf(-\infty,t) \cap T. \end{cases},$$

The point $t \in T$ is left-dense, left-scattered, right-dense, right-scattered if

$$\rho(t) = t,\ \rho(t) < t,\ \sigma(t) = t,\ \sigma(t) > t,$$

respectively. The graininess function μ, for a time scale, is defined by:

$$\mu(t) = \sigma(t) - t.$$

For a function $\phi: T \to R$, the function ϕ^σ denotes $\phi \circ \sigma$. The Δ -derivative of $\phi: T \to R$ at point t is defined by:

$$\phi^{\Delta}(t)=\begin{cases} \lim_{s\to t}\dfrac{\phi(t)-\phi(s)}{t-s}, & if t is right dense,\\ \dfrac{\phi^{\sigma}(t)-\phi(t)}{\mu(t)}, & if t is right scattered,\end{cases}$$

A function $\phi: T\to R$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points. The set of rd-continuous functions $\phi: T\to R$ is denoted by $C_{rd}(T,R)$.

We introduce the following notations that will be used throughout the paper :

$$[a,b]_T=[a,b]\cap T,$$
$$and$$
$$T^{\kappa}=\overline{T-\{sup\, T\}}.$$

Definition 1 [11]A function $\Phi: [a,b]_T\to R$ is called a delta antiderivative function $\phi: [a,b]_T\to R$ provided Φ is continuous on $[a,b]_T$, delta differentiable on $[a,b)$, and $\Phi^{\Delta}(t)=\phi(t)$, for all $t\in[a,b)$. Then, we define the Δ -integral of ϕ from a to b by:

$$\int_a^b \phi(t)\Delta t=\Phi(b)-\Phi(a),$$

we write for a function ϕ is integrable on $[a,b]_T$ by:

$$\phi\in L^1_{\Delta}([a,b]_T,R).$$

Proposition 1 [1]Let $\phi\in L^1_{\Delta}([a,b]_T,R)$. Then

$$\int_a^b \phi(t)\Delta t=\int_{[a,b]_T}\phi(t)dt+\sum_{t\in R\cap[a,b)}\mu(t)\phi(t),$$

where

$$R=\{t\in T: \sigma(t)>t\},$$

is at most countable.

1.2. Some Properties of Function Spaces on Time Scales

We also define the following function spaces:

$$C^1(T,R)=\{\phi: T\to R: \phi\, is \Delta-differentiable\ on\ T^{\kappa}, and\ \phi^{\Delta}\in C(T^{\kappa},R)\},$$

$$C^1_{rd}(T,R)=\{\phi: T\to R: \phi\, is \Delta-differentiable\ on\ T^{\kappa}, and\ \phi^{\Delta}\in C_{rd}(T^{\kappa},R)\}.$$

Definition 2 [2]Let $p\in 1,+\infty)$. The space $L^p_{\Delta}(T,R)$ denotes the set of functions $\phi: T\to R$ such that

$$P\phi P^p_{L^p_{\Delta}(T,R)}=\int_{[a,b)\cap T}|\phi(s)|^p\Delta s<\infty,$$

where Δ denotes the *delta integral* on the time scale T. Equipped with this norm, $L^p_{\Delta}(T,R)$ is a Banach space.

The diagram below presents the density result between functional spaces on time scales, as demonstrated by Benaissa *et al.* [2] of the referenced literature,

$$\begin{array}{ccccc} C_{rd}(T,R) & \rightarrow & L^p_\Delta(T,R) & \rightarrow & L^1_\Delta(T,R) \\ & & \uparrow & & \\ & & C(T,R) & & \\ & & \uparrow & & \\ & & C^1_{rd}(T,R) & & \end{array}$$

Definition 3 [1]Let $p \in 1,+\infty)$. Define the Sobolev-type space $W^{1,p}_\Delta(T,R)$ as the set of all $\phi \in L^p_\Delta(T,R)$ such that $\phi^\Delta \in L^p_\Delta(T,R)$, with the norm:

$$P\phi P_{W^{1,p}_\Delta(T,R)} = P\phi P_{L^p_\Delta(T,R)} + P\phi^\Delta P_{L^p_\Delta(T,R)}.$$

Then, $W^{1,p}_\Delta(T,R)$ is a Banach space.

The diagram below illustrates the density result between functional spaces on time scales, as established by Benaissa *et al.* [1]. This result shows that the space of smooth functions with compact support is dense in Sobolev spaces on arbitrary time scales. Such a density property is a cornerstone in the theory of functional spaces, as it allows the approximation of Sobolev functions by smoother, more regular ones. This is crucial for both theoretical analysis and the development of numerical methods within the time scale framework.

$$C^1_{rd}(T,R) \quad \rightarrow \quad W^{1,p}_\Delta(T,R) \quad \rightarrow \quad L^p_\Delta(T,R).$$

1.3. Riemann–Liouville Fractional on Time Scales

We recall that Benkhettou *et al.* [13] established existence and uniqueness results for Riemann–Liouville fractional equations on time scales.

Definition 4 [13]Let $\phi \in L^1_\Delta([a,b]_T,R)$ and $0 < \alpha < 1$. Then the (left) fractional integral of order α of ϕ is defined by:

$${}^T_aI^\alpha_t\phi(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s)\Delta s,$$

where Γ is the gamma function.

Definition 5 [13]Let $t \in T$, $0 < \alpha < 1$, and $\phi: T \to R$. The (left) Riemann–Liouville fractional derivative of order α of ϕ is defined by:

$${}^T_aD^\alpha_t\phi(t) = \left(\int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\phi(s)\Delta s\right)^\Delta = \left({}^T_aI^{1-\alpha}_t\phi(t)\right)^\Delta.$$

This party highlights the main contributions that exist in [8], which provides several corrections and improvements to the theory of fractional derivatives on time scales.

Definition 6 (Beta function on time scales) [8]We will define the function $B^T_{a,b}(\alpha,\beta)$ as follows

$$B^T_{a,b}(\alpha,\beta) = \int_a^b (s-a)^{\beta-1}(b-s)^{\alpha-1}\Delta s, \quad \text{for} \alpha > 0 \quad \text{and} \quad \beta > 0,$$

where

$$T_{a,b} = \frac{1}{b-a}(T-a).$$

Lemma 1 [8]Let $\phi \in L_{\Delta}^{1}([a,b]_{T},R)$, the Riemann–Liouville Δ -fractional integral satisfies

$$\left({}_{a}^{T}I_{t}^{\beta} \circ {}_{a}^{T}I_{t}^{\alpha}\right)(\phi(t)) = \frac{1}{\Gamma(\alpha+\beta)}\int_{a}^{t} \phi(u)(t-u)^{\beta+\alpha-1}\frac{\beta_{0,1}^{T_{u,t}}(\beta,\alpha)}{B(\alpha,\beta)}\Delta u,$$

for $\alpha > 0$ and $\beta > 0$.

2. Main Results

Now, we introduce and recall a new notion of L_{Δ}^{p}-representability adapted to the time scale framework, which extends the classical concept to a more general setting.

Lemma 2 Let $p \in 1,+\infty)$ and $\alpha \in (0,1)$, such that $\alpha - \frac{1}{p} > 0$, then

$${}_{a}^{T}I_{t}^{\alpha}\left(L_{\Delta}^{p}(T,R)\right) \subset C_{rd}(T,R),$$

and

$$\| {}_{a}^{T}I_{t}^{\alpha}\phi\|_{\infty} \leq c(T,\alpha,p)P\phi P_{L_{\Delta}^{p}(T,R)}, \quad for\ \phi \in L_{\Delta}^{p}(T,R),$$

where

$$c(T,\alpha,p) = \frac{(b-a)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(q(\alpha-1)+1)^{\frac{p-1}{p}}} > 0.$$

Proof. Let $\phi \in L_{\Delta}^{p}(T,R)$, by inequality of Hölder, we have

$$| {}_{a}^{T}I_{t}^{\alpha}\phi(t)| \leq \int_{a}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\phi(s)|\Delta s$$

$$\leq \frac{1}{\Gamma(\alpha)}\left(\int_{a}^{t} (t-s)^{q(\alpha-1)}\Delta s\right)^{\frac{1}{q}} P\phi P_{L_{\Delta}^{p}(T,R)}$$

$$\leq \frac{1}{\Gamma(\alpha)}\left(\int_{a}^{t} (t-s)^{q(\alpha-1)}ds\right)^{\frac{1}{q}} P\phi P_{L_{\Delta}^{p}(T,R)}$$

$$\leq \frac{(t-a)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(q(\alpha-1)+1)^{\frac{p-1}{p}}} P\phi P_{L_{\Delta}^{p}(T,R)}.$$

Additionally, it is necessary.

Remark 1 Let $p \in 1,+\infty)$ and $\alpha \in (0,1)$, such that $\alpha - \frac{1}{p} > 0$, by lemma 2, we can define the space $I_{T}^{\alpha}\left(L_{\Delta}^{p}(T,R)\right)$ as follows

$$I_{T}^{\alpha}\left(L_{\Delta}^{p}(T,R)\right) = \left\{\phi: T \to R: \exists\psi \in L_{\Delta}^{p}(T,R): {}_{a}^{T}I_{T}^{\alpha}\psi(t) = \phi(t)\right\},$$

Definition 7 Let $q \in 1,+\infty)$, we say that ϕ is L_{Δ}^{p}-representable, if $\phi \in I_{T}^{\alpha}\left(L_{\Delta}^{p}(T,R)\right)$, for some $1 \leq p \leq q$ and $\alpha \in (0,1)$.

Lemma 3 Let $q \in 1,+\infty)$ and $\alpha \in (0,1)$, such that $\alpha - \frac{1}{p} > 0$, then

$$I_T^{\alpha}\left(L_{\Delta}^{p}(T,R)\right)^{\circ}L_{\Delta}^{p}(T,R).$$

Proof. Let $\phi \in I_T^{\alpha}\left(L_{\Delta}^{p}(T,R)\right)$, then there is $\psi \in L_{\Delta}^{p}(T,R)$, such that ${}_{a}^{T}I_T^{\alpha}\psi(t) = \phi(t)$,

$$\|\phi\|_{I_T^{\alpha}(L_{\Delta}^{p}(T,R))} = \|\psi\|_{L_{\Delta}^{p}(T,R)},$$

by Lemma 2, we obtain

$$\|\phi\|_{I_T^{\alpha}(L_{\Delta}^{p}(T,R))} \leq \|\phi\|_{L_{\Delta}^{p}(T,R)},$$

which means

$$I_T^{\alpha}\left(L_{\Delta}^{p}(T,R)\right)^{\circ}L_{\Delta}^{p}(T,R).$$

The subsequent lemma establishes a characterization of L_{Δ}^{p}-representability. For conciseness, we present and prove it in a particular case, noting that other cases can be treated analogously.

Lemma 4 Let $q \in 1,+\infty)$, $\phi \in L_{\Delta}^{p}(T,R)$, $\alpha \in (0,1)$ and $1 \leq p \leq q$, we have that

$$\phi \in I_T^{\alpha}\left(L_{\Delta}^{p}(T,R)\right),$$

if and only if

$${}_{a}^{T}I_t^{1-\alpha}\phi \in W_{\Delta}^{1,p}(T,R),$$

and

$${}_{a}^{T}I_t^{1-\alpha}\phi(a) = 0.$$

Proof. Let $\phi \in I_T^{\alpha}\left(L_{\Delta}^{p}(T,R)\right)$, then there is $\psi \in L_{\Delta}^{p}(T,R)$, such that ${}_{a}^{T}I_T^{\alpha}\psi(t) = \phi(t)$. Lemma 1 gives us

$${}_{a}^{T}I_t^{1-\alpha}\phi(t) = {}_{a}^{T}I_t^{1-\alpha}\left({}_{a}^{T}I_T^{\alpha}\psi(t)\right)$$

$$= \int_a^t \psi(u)\frac{\beta_{0,1}^{T_{u,t}}(1-\alpha,\alpha)}{B(1-\alpha,\alpha)}\Delta u,$$

Since the function

$$(u,t) \to \frac{\beta_{0,1}^{T_{u,t}}(1-\alpha,\alpha)}{B(1-\alpha,\alpha)},$$

is continuous on $T \times T$, then

$${}_{a}^{T}I_t^{1-\alpha}\phi \in W_{\Delta}^{1,p}(T,R),$$

and

$${}_{a}^{T}I_t^{1-\alpha}\phi(a) = 0.$$

On the other hand, if

$$ {}_{a}^{T}I_{t}^{1-\alpha}\phi \in W_{\Delta}^{1,p}(T,R), $$

and

$$ {}_{a}^{T}I_{t}^{1-\alpha}\phi(a) = 0, $$

as

$$ W_{\Delta}^{1,p}(T,R)^{\circ}C_{rd}(T,R), $$

so

$$ \int_{a}^{t} \left({}_{a}^{T}D_{t}^{\alpha}\phi(s)\right)\Delta s = \int_{a}^{t} \left({}_{a}^{T}I_{t}^{1-\alpha}\phi(s)\right)^{\Delta}\Delta s = {}_{a}^{T} I_{t}^{1-\alpha}\phi(t), $$

as

$$ {}_{a}^{T}I_{t}^{1-\alpha}\phi \in W_{\Delta}^{1,p}(T,R), $$

then

$$ \left({}_{a}^{T}I_{t}^{1-\alpha}\phi\right)^{\Delta} \in L_{\Delta}^{p}(T,R), $$

consequently

$$ {}_{a}^{T}I_{t}^{1-\alpha}\phi \in L_{\Delta}^{p}(T,R), $$

what is means

$$ {}_{a}^{T}I_{T}^{\alpha}\left({}_{a}^{T}I_{t}^{1-\alpha}\phi\right) \in I_{T}^{\alpha}\left(L_{\Delta}^{p}(T,R)\right), $$

by Lemma 1, we have

$$ \left({}_{a}^{T}I_{t}^{1-\alpha} \circ {}_{a}^{T} I_{t}^{\alpha}\right)|\phi| \geq {}_{a}^{T} I_{t}^{1}|\phi|, $$

therefore

$$ {}_{a}^{T}I_{t}^{1}|\phi| \in I_{T}^{\alpha}\left(L_{\Delta}^{p}(T,R)\right). $$

From this, we conclude that

$$ \phi \in I_{T}^{\alpha}\left(L_{\Delta}^{p}(T,R)\right). $$

We are now in a position to introduce the left Riemann–Liouville fractional Sobolev spaces on time scales, which will be formally defined below.

Definition 8 (Riemann-Liouville Fractional Sobolev Spaces) Let $p \in 1,\infty]$ and $\alpha \in (0,1)$. The left Riemann-Liouville fractional Sobolev space of order α and summability p is defined as

$$W^{\alpha,p}_{RL,a^+}(T,R) = \{\phi \in L^p(T,R) : {}^T_a I^{1-\alpha}_t(\phi) \in W^{1,p}_\Delta(T,R)\},$$

where ${}^T_a I^{1-\alpha}_t(\phi)$ denotes the left Riemann-Liouville fractional integral of order $1-\alpha$ of the function ϕ starting from a.

Given that the spaces $L^p(T,R)$ and $W^{1,p}_\Delta(T,R)$ are white spaces, the following remark naturally follows and will be instrumental in the developments that follow.

Remark 2 It is not difficult to see that the space $W^{\alpha,p}_{RL,a^+}(T,R)$, endowed with the norm

$$\|\phi\|_{W^{\alpha,p}_{RL,a^+}(T,R)} = \|\phi\|_{L^p(T,R)} + \| \; {}^T_a I^{1-\alpha}_t(\phi)\|_{W^{1,p}_\Delta(T,R)} .$$

is a Banach space.

Lemma 5 Let $\alpha \in (0,1)$ and $p \in [1,\infty)$. Then

$$\phi \in I^\alpha_T\big(L^p_\Delta(T,R)\big),$$

if and only if

$$\phi \in W^{\alpha,p}_{RL,a^+}(T,R),$$

and

$${}^T_a I^{1-\alpha}_t \phi(a) = 0.$$

Proof. In the light of Definition 7, we may restate Lemma 4 in an equivalent form that aligns more naturally with the structure under consideration.

3. Conclusion

In this paper, we introduced and analyzed a new class of fractional Sobolev spaces on time scales, associated with the left Riemann–Liouville fractional integral. We established fundamental properties of the space $W^{\alpha,p}_{RL,a^+}(T,R)$, including its topological structure and its relationship to classical function spaces on time scales. This space provides a natural framework for the study of fractional dynamic equations involving nonlocal operators.

Moreover, the results obtained here can be readily extended to the right Riemann–Liouville case, as the underlying techniques remain valid with minor modifications. The importance of such functional spaces lies in their applicability to the well-posedness and analysis of fractional differential equations on time scales.

Finally, the methods and constructions presented in this work may also be generalized to the nabla-type fractional setting, which opens new directions for future research in fractional time scale calculus and its applications.

Conflict of Interest

The authors have no relevant financial or non-financial interests to disclose.

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